Lecture 5 - Free groups, functors

Note Title 1/31/2008

<u>Prop 18</u> 1) Any cyclic group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ some n.

2) A subgroup of a cyclic group is cyclic

<u>Remark</u>: Any subgroup of $\mathbb{Z}$ is of the form $m\mathbb{Z} = \{mn \mid n \in \mathbb{Z}\}$ for some m.

Use the division alg: $k \in H \subseteq \mathbb{Z}$, pick m to be the smallest positive element of H, then

$$\underset{H}{\overset{\Uparrow}{k}} = q\underset{H}{\overset{\Uparrow}{m}} + r \Rightarrow \underset{H}{\overset{\Uparrow}{k - qm}} = \underset{H}{\overset{\Uparrow}{r}}$$

$$0 \le r \le m-1 \Rightarrow r = 0 \Rightarrow k = qm.$$

<u>Pf of Prop 18</u>: Pick a generator of G: g. Such a choice give

$$g: \mathbb{Z} \longrightarrow G \qquad \text{(a hom b/c of Lemma 3)}$$
$$n \longmapsto g^n$$

g is a surjective map. (everything in G is of the form g^m some m)

1) 1^{st} Iso thm: $G \cong \mathbb{Z}/\ker(g) = \mathbb{Z}/m\mathbb{Z}$ some m

$m = o(g) =$ smallest power that takes us to e.

$$g^m = e, \quad g^{<m} \neq e$$

2) Cor. Thm: Subgroups of $G \longleftrightarrow$ subgroups of $\mathbb{Z}$ containing $m\mathbb{Z}$.

$=$ subgroups of the form $k\mathbb{Z}$, some $k \mid m$

(pick the generator of the bigger subgroup k, then $m \in m\mathbb{Z} \subseteq k\mathbb{Z}$ is of the form $m = k \cdot q \Rightarrow k \mid m$)

Subgroups of G are cyclic and of the form

$$k\mathbb{Z}/m\mathbb{Z}, \quad k \mid m$$
$$\cong$$
$$\mathbb{Z}/q\mathbb{Z}, \quad m = kq \qquad \blacksquare$$

Def Given a set S, the free group on S, $F(S)$ is the set of all "words" of the form

$a_1 \cdots a_n$ where a_i or $a_i^{-1} \in S$, and n varies.

Mult is concatenation subject to canceling a and a^{-1}.

ie $a_1 \cdots a_r a a^{-1} a_{r+1} \cdots a_n = a_1 \cdots a_r a_{r+1} \cdots a_n$

unit = empty string

$= aa^{-1} = a^{-1}a$ for any $a \in S \cup S^{-1}$.

Key Property: $\mathrm{Hom}_{Gps}(F(S), G) = \mathrm{Hom}_{Set}(S, \underset{\uparrow}{U}(G))$ underlying set.

If $S = \{s_1, \ldots, s_n\}$ $G = (U(G), \cdot)$

$$\begin{array}{l} f: S \to U(G) = G \\ \quad s_i \mapsto g_i \end{array} \implies \begin{array}{l} \tilde{f}: F(S) \to G \\ \quad a_1 \cdots a_n \longmapsto f(a_1) \cdot f(a_2) \cdot \ldots \cdot f(a_n) \\ \quad s_i^{-1} \longmapsto g_i^{-1} \end{array}$$

Given
$$\begin{tikzcd} F(S) \arrow[r, "\tilde{f}"] & G \\ S \arrow[u, phantom, "\cup"] \arrow[ur, "f"'] & \end{tikzcd}$$

Ex $F(\{x,y\}) = \{ \_, x, y, x^{-1}, y^{-1}, x^2, xy, yx, y^2, xy^{-1}, x^{-1}y, yx^{-1}, y^{-1}x, y^{-2}, x^{-2}, x^{-1}y^{-1}, y^{-1}x^{-1} \ldots \}$

$\mathrm{Hom}_{Grp}(F(\{x,y\}), G) = \mathrm{Hom}_{Set}(\{x,y\}, G)$

$f \longleftrightarrow$ pick 2 elements of G $f(x), f(y)$ / $g \quad h$

$\tilde{f}(x^{-1}y) = f(x)^{-1} \cdot f(y) = g^{-1}h$
$= \tilde{f}(x^{-1}) \cdot \tilde{f}(y)$

$\mathrm{Im}(\tilde{f}) = \langle g, h \rangle$

$F(\{x\}) \cong \mathbb{Z}$

$x \longleftrightarrow 1$

Morphisms in $\underline{Cat}$

C, D are categories.

Def A functor from C to D is a pair of functions:

$$Obj_C \xrightarrow{F} Obj_D$$
$$Mor_C \xrightarrow{F} Mor_D$$

$$F: Hom_C(a,b) \longrightarrow Hom_D(F(a), F(b))$$
$$F(f \circ g) = F(f) \circ F(g)$$
$$F(1_a) = 1_{F(a)}$$

Ex: $\mathcal{U}: \mathcal{G}ps \longrightarrow \mathcal{S}ets$ $\quad$ $\mathcal{U}$ is a forgetful functor

$(G, \cdot) \longmapsto G$ $\quad$ $\leftrightarrow$ forget the mult.

$(f: (G, \cdot_G) \to (H, \cdot_H)) \longmapsto f: G \longrightarrow H$

Any group is a category:

$\underline{G}$: $\quad Obj = \{ * \}$

$Mor = G$

$g, h \in G = Hom(*, *)$, then comp is mult in the group.

$e \in G$ is 1_*

A functor $\underline{G} \xrightarrow{F} \underline{H}$ $\longleftrightarrow$ a homomorphism $G \to H$

$\Updownarrow$

together make F $\begin{cases} F: \{*\} \to \{*\} \\ F: G \to H \end{cases}$

that preserves the comp $= \text{mult.}$

Ex: $F: \mathcal{S}ets \longrightarrow \mathcal{G}rps$

$S \longmapsto F(S)$

$f: S \longrightarrow T \subseteq F(T)$ $\quad$ Given $f: S \to T$, get $f: S \to F(T)$

$$Hom_{Sets}(S, \underset{G}{F(T)}) = Hom_{Grp}(F(S), \underset{G}{F(T)})$$

$$\Rightarrow F(f): F(S) \longrightarrow F(T)$$
$$a_1 \cdots a_n \longmapsto f(a_1) \cdots f(a_n)$$

F and U are called adjoint functors:

$$\mathrm{Hom}_{Set}(S, U(G)) = \mathrm{Hom}_{Grp}(F(S), G)$$

Ex: G be a group, A be an abelian group

$f: G \longrightarrow A$ be a homomorphism

$$g, h \in G \Rightarrow f(g)\cdot f(h) = f(h)\cdot f(g)$$

$$\Rightarrow \underbrace{f(g)\cdot f(h)\cdot f(g)^{-1}\cdot f(h)^{-1}}_{= f(ghg^{-1}h^{-1})} = e$$

$$\Rightarrow \underbrace{ghg^{-1}h^{-1}}_{=[g,h]} \in \ker(f)$$

$[g,h]$ = commutator of g & h

Def The commutator subgroup of G is the s.g. gen by $[g,h]$ $\forall g, h \in G$.

$G' = [G,G]$

Hw: $[G,G]$ is a normal s.g & $G/[G,G]$ is abelian

The assignment $G \longmapsto G/[G,G]$ is a functor

$\mathcal{G}rps \longrightarrow Ab\mathcal{G}rps$ "abelianization"

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr] & H \arrow[d] \\ & H/[H,H] \end{tikzcd}$$

$\Rightarrow \tilde{f}: G \longrightarrow H/[H,H]$

$[G,G] \subseteq \ker(\tilde{f}) \Rightarrow \tilde{f}$ factors uniquely as

$$\begin{tikzcd} G \arrow[d] \arrow[dr] & \\ G/[G,G] \arrow[r, "\hat{f}"'] & H/[H,H] \end{tikzcd}$$

Def If the abelianization of G is trivial $= \{e\}$, then G is perfect.

If G is a group, then $U(G)$ is a set, and the identity map

is a nice function in $\mathrm{Hom}_{Sets}(U(G), U(G))$
$$\| $$
$$\mathrm{Hom}_{Grps}(F(U(G)), G)$$

$$\begin{array}{l} F(U(G)) \\ \quad \| \\ F(G) \longrightarrow G \\ \ \ \cup \\ \ \ G \end{array}$$

$F(U(G)) \longrightarrow G$ is surjective, so
$G \cong F(U(G))/\ker$

Such a description is a "presentation".

If S is a set of generators for G, then
$$\begin{array}{ccc} F(S) & \longrightarrow & G \\ s & \longmapsto & s \end{array}$$
is surjective, and

$$G = F(S)/\ker$$
$\uparrow$ generators here are relations.

$G = \langle S \rangle$ $\quad$ S as a subset
$= \langle S \mid R \rangle$
$\uparrow$ gen of $\ker$.

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} = \{x, y \mid x^4, y^2 = x^2, xyx^{-1} = y^{-1}\}$$
$ij = k$
$i^2 = j^2 = k^2 = -1$