Lecture 4 - Isomorphism Thms

Note Title 1/29/2008

Ended w/ $G \xrightarrow{\pi_N} G/N$ is a group hom. if N is normal.

<u>Def</u>: The kernel of a map $f: G \to H$ is

$$\ker(f) = \{ g \in G \mid f(g) = e_H \}$$

The image of f is all points in H hit by f:

$$\mathrm{Im}(f) = \{ h \in H \mid \exists g \in G,\ f(g) = h \}$$

<u>Prop 11</u> For any f, $\ker(f)$ is a normal s.g & $\mathrm{Im}(f)$ is a subgroup of H.

Pf: $\ker(f) \underset{\text{normal}}{\triangleleft} G$: 1) $\ker(f)$ is a subgroup

2) $g^{-1}kg \in \ker(f)$ $\forall g \in G$, $k \in \ker(f)$

1) $e_G \in \ker(f) \Rightarrow$ non-empty. If $g, h \in \ker(f)$, then $f(g) = f(h) = e$

$$g^{-1}h \overset{?}{\in} \ker(f) \longleftrightarrow f(g^{-1}h) \overset{?}{=} e$$

$$f(g^{-1}) \cdot f(h) = \underbrace{(f(g))^{-1}}_{e_H} \cdot \underbrace{f(h)}_{e_H} = e_H$$

2) If $f(k) = e$, then

$$\begin{aligned} f(g^{-1}kg) &= (f(g))^{-1} f(k) f(g) \\ &= (f(g))^{-1} \cdot e_H \cdot f(g) \\ &= (f(g))^{-1} \cdot f(g) \\ &= e_H \end{aligned}$$

$$\Rightarrow g^{-1}kg \in \ker(f).$$

for any $f: G \to H$, $G/\ker(f)$ is a group (coimage)

<u>Thm 12</u> (1^{st} Isom Theorem) If $f: G \to H$, then $G/\ker(f) \cong \mathrm{Im}(f)$.

isomorphic = bijective homomorphism

<u>Pf</u>: Define $\tilde{f}: G/\ker(f) \to H$ by $\tilde{f}([a]) = f(a)$

1) $\tilde{f}$ is well-defined

2) $\tilde{f}$ is a homomorphism

3) $\tilde{f}$ is a bijection

1) follows if $f(a\cdot k) \overset{?}{=} f(a)$ for any $k \in \ker(f)$

$$\begin{matrix} f(a\cdot k) & \overset{?}{=} & f(a) \\ \| & & \| \\ f(a)\cdot f(k) & = & f(a)\cdot e_H \end{matrix}$$

2) $$\begin{matrix} \tilde f([a]\cdot[b]) & \overset{?}{=} & \tilde f([a])\cdot \tilde f([b]) \\ \| & & \| \\ \tilde f([ab]) & & f(a)\cdot f(b) \\ \| & & \\ f(a\cdot b) & = & \end{matrix}$$

3) $\mathrm{Im}(\tilde f) = \mathrm{Im}(f)$. $\qquad h \in \mathrm{Im}(f) \leftrightarrow \underset{\text{some } g\in G}{h = f(g)} = \tilde f([g])$

$\Leftrightarrow h \in \mathrm{Im}(\tilde f)$

$[g] \in \ker(\tilde f)$, $\leftrightarrow \underset{f(g)}{\underset{\|}{\tilde f([g])}} = e_H \Leftrightarrow g \in \ker(f) \leftrightarrow [g] = [e]$

$\Rightarrow \tilde f$ is a bijective homomorphism $G/\ker(f) \longrightarrow \mathrm{Im}(f)$. $\boxtimes$

f is injective iff $\ker(f) = \{e\}$

$f(a) = f(b) \Leftrightarrow f(a)\cdot f(b)^{-1} = e \Leftrightarrow f(a)\cdot f(b^{-1}) = e \Leftrightarrow f(ab^{-1}) = e$

$\Leftrightarrow ab^{-1} \in \ker(f)$

Thm 13 If $N \trianglelefteq G$, and f is any homomorphism $G \to H$ s.t. $N \subseteq \ker(f)$, then there is a unique homomorphism

$G/N \xrightarrow{\tilde f} H$ s.t.

$$\begin{tikzcd} G \arrow[d, "\pi_N"'] \arrow[dr, "f"] \arrow[dr, phantom, bend right=20, "\circlearrowleft"] & \\ G/N \arrow[r, "\tilde f"'] & H \end{tikzcd}$$

Pf: $\tilde f([a]) = f(a)$. Argument for Thm 12 implies that $\tilde f$ is well defined & a homomorphism.

Prop 14 If N is a normal s.g of G, and H is a s.g, then

$HN = \{h\cdot n \mid h \in H, n \in N\}$ is a s.g of G

$\|$

$NH = \{n\cdot h \mid \quad " \quad \}$

Pf: N normal $\Rightarrow h^{-1} n h \in N \quad \forall h \in H$

$h^{-1} n h = n_1$ some $n_1 \Leftrightarrow n\cdot h = h\cdot n_1$

$(N\cdot h = h\cdot N)$

$$\underset{\in HN}{(h\cdot n)^{-1}} = \underset{\in N}{n^{-1}}\cdot \underset{\in H}{h^{-1}} = \underset{\in HN}{h^{-1}\cdot n_1''}$$

$$\underset{\in HN}{(hn)}\underset{\in HN}{(km)} = h(nk)m = h(k\cdot n')m = \underset{\in H}{(hk)}\cdot\underset{\in N}{(n'm)} \qquad \square$$

Thm 15 If $N \trianglelefteq G$, $H \leq G$, then

$$H/_{H\cap N} \cong HN/_N$$

Pf:

$$\begin{tikzcd} G \arrow[r, "\pi_N"] & G/N \\ H \arrow[u, phantom, "\cup"] \arrow[ur, "\pi_N"'] & \end{tikzcd}$$

Can restrict π_N to H

$$\begin{aligned} \ker(\pi_N|_H) &= \{h\in H \mid \pi_N(h) = [e]\} \\ &= \{h \in H \mid h \in \ker(\pi_N)\} \\ &= \{h\in H \mid h \in N\} \\ &= H\cap N \end{aligned}$$

Thm 12 $\Rightarrow$ $H/_{H\cap N} \cong \operatorname{Im}(\pi_N|_H) = HN/_N$

$$\{[h] \mid h\in H\} = \{[h\cdot n] \mid h\in H, n\in N\} \qquad \square$$

Thm 16 If $N \trianglelefteq G$, $H \trianglelefteq G$, $H \supseteq N$, then

$$(G/_N)/(H/_N) \cong G/_H$$

Pf By Thm 13, have a homomorphism

$$\begin{tikzcd} G/N \arrow[r, "\pi"] & G/H \\ G \arrow[u, "\pi_N"] \arrow[ur, "\pi_H"'] & \end{tikzcd} \qquad (H = \ker(\pi_H) \supseteq N)$$

$\pi(aN) = aH$

$$\begin{aligned} \ker(\pi) &= \{aN \mid aH = H \longleftrightarrow a\in H\} = \{hN, h\in H\} \\ &= H/N \end{aligned}$$

$\operatorname{Im}(\pi) = G/H$

$\Rightarrow$ Thm 12: $(G/N)/_{(H/N)} \cong G/_H$ $\qquad \square$

Thm 17 If $N \trianglelefteq G$, there is a 1-1, onto, order preserving map between subgroups of G containing N & subgroups of G/N.

Pf: $(H \geq N) \longleftrightarrow H/N$

1-1) $H_1, H_2 \geq N$ & $H_1/N = H_2/N$

If $h_1 \in H_1$, then $h_1 N = h_2 N$ for some $h_2 \in H_2$

In particular $\underset{h_1}{\underset{\shortparallel}{h_1 \cdot e}} = \underset{H_2}{\underset{\cap}{h_2}} \cdot \underset{H_2}{\underset{\cap}{n}} \in H_2 \Rightarrow H_1 \subseteq H_2$

swap H_1 & H_2 everywhere

$\Rightarrow H_2 \subseteq H_1 \Rightarrow H_1 = H_2$

onto) for any $H' \leq G/N$, $\pi_N^{-1}(H') = \{ g \mid \pi_N(g) \in H' \}$ is a subgroup of G, containing N.

$g, h \in \pi_N^{-1}(H')$, then $\pi_N(g^{-1}h) \overset{?}{\in} H' \longleftrightarrow g^{-1}h \in \pi_N^{-1}(H')$

$\updownarrow$

$[g] \in H'$, $[h] \in H'$ $\Longrightarrow$ $\pi_N(g^{-1}h) = [g^{-1}] \cdot [h] = [g]^{-1} \cdot [h] \in H'$

$\pi_N(\pi_N^{-1}(H')) = H'$ $\square$