Lecture the Last – Jordan Form

Note Title 4/15/2008

Applying structure theory to $k[x]$-modules.

<u>Thm</u> If M is a f.g. torsion $k[x]$-module, then we can find polynomials $f_1(x), \dots, f_k(x)$ s.t.

$$M \cong k[x]/(f_1(x)) \oplus \dots \oplus k[x]/(f_k(x))$$

& $f_1(x) \mid f_2(x) \dots \mid f_k(x)$.

M is uniquely determined by the sequence of monic poly $f_1(x) \mid f_2(x) \mid \dots \mid f_k(x)$ leading coef $= 1$.

<u>Def</u> . $f_k(x)$ gen the $\mathrm{Ann}(M)$ and is called the <u>minimal polynomial</u>

. $f_1(x) \cdots f_k(x)$ is the <u>characteristic poly of M</u>. $p_M(x)$

<u>Ex</u> . If $p_M(x) = (x-a_1)\cdots(x-a_n)$ a_i all distinct, then there is only one possible M:

$$M = k[x]/p_M(x).$$

. If $p_M(x) = (x-1)^2 \cdot x$

x occurs in last invariant factor

- $f_1(x) = (x-1)$, $f_2(x) = x(x-1)$ $\rightsquigarrow$ $M = \left(k[x]/_{x-1}\right) \oplus k[x]/x(x-1)$
- $f_1(x) = x(x-1)^2$ $\rightsquigarrow$ $M = k[x]/x(x-1)^2$

<u>Remark</u> We can run our primary decomp as well:

Let $p_1(x) \dots p_r(x)$ be the primes dividing the minimal poly (char poly)

Then $M \cong M_{p_1(x)} \oplus \dots \oplus M_{p_r(x)}$

$$M_p = \{ m \in M \mid p^k . m = 0 \text{ some } k \}$$

<u>Ex</u> $p_M(x) = x(x-1)^2$:

$$M \cong M_x \oplus M_{(x-1)}$$

"part w/ $p_M(x) = x$" "part w/ $p_M(x) = (x-1)^2$"

$\overset{\shortparallel}{k[x]/x}$ $\oplus$ $\begin{cases} k[x]/_{x-1} \oplus k[x]/_{x-1} & f_1 = f_2 = x-1 \\ k[x]/_{(x-1)^2} & f_1 = (x-1)^2 \end{cases}$

Let V be a f.d. k-vector space.

Let $T \in \mathrm{End}_k(V)$: $T: V \to V$

$\longleftrightarrow$ V is a $k[x]$-module via

$$p(x) \cdot \bar{v} = p(T)(\bar{v})$$

Ex $p(x) = x-3$, then $p(x).\bar{v} = (T-3)(\bar{v}) = T(\bar{v}) - 3\bar{v}$

A basis for V is a gen set as a $k[x]$-mod.

Let $\bar{v} \in V$, $\mathrm{Ann}(\bar{v}) \neq \{\bar{0}\}$.

Consider the set $\{\bar{v}, T(\bar{v}), T^2(\bar{v}), \dots\}$

Since V is f.d, know there is some n s.t.

$\{\bar{v}, T(\bar{v}), \dots, T^n(\bar{v})\}$ is lin. dep. & $\{\bar{v}, T(\bar{v}), \dots, T^{n-1}(\bar{v})\}$ is ind.

$\Rightarrow$ can find $a_0, \dots, a_{n-1}$ s.t. $a_0\bar{v} + \dots + a_{n-1}T^{n-1}\bar{v} + T^n(\bar{v}) = \bar{0}$

$\Rightarrow$ $x^n + a_{n-1}x^{n-1} + \dots + a_0 \in \mathrm{Ann}(\bar{v})$.

If $\{\bar{v}_1, \dots, \bar{v}_m\}$ is a basis for V, then

$\mathrm{Ann}(V) = \mathrm{Ann}(\bar{v}_1) \cap \dots \cap \mathrm{Ann}(\bar{v}_m) =$ ideal gen by the lcm of the gen of $\mathrm{Ann}(\bar{v}_i)$.

$= \langle m(x) \rangle$

minimal polynomial.

The "Normal Form" / "Rational Form" is V decomposed into invariant factors:

$$V = k[x]/f_1(x) \oplus k[x]/f_2(x) \oplus \dots \oplus k[x]/f_s(x).$$

$k[x]/f(x) = x^n + a_{n-1}x^{n-1} + \dots$ has a k-basis given by

$\Rightarrow x^n = -a_{n-1}x^{n-1} - \dots - a_0$ $\quad 1, x, x^2, \dots, x^{n-1}$

& $[T]_{\{1, x, \dots x^{n-1}\}} =$

$$\begin{array}{c|c} & \begin{matrix} 1 & x & x^2 & \cdots & & x^{n-1} \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2 \\ \vdots \\ \\ x^{n-1} \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots 0 & -a_0 \\ 1 & 0 & 0 & & & -a_1 \\ 0 & 1 & 0 & & & \vdots \\ \vdots & \vdots & 1 & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots 1 & -a_{n-1} \end{bmatrix} \end{array}$$

$$T = \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{array} \quad \text{acting on } \mathbb{R}^3$$

$T(e_3) = 2e_3 : \quad \mathrm{Ann}(\bar{e}_3) = \langle x-2 \rangle$

$e_1,\ T(e_1) = e_2,\ T^2(e_1) = e_1 \quad \Rightarrow \quad \mathrm{Ann}(\bar{e}_1) = \langle x^2-1 \rangle$

$e_2,\ T(e_2) = e_1,\ T^2(e_2) = e_2 \quad \Rightarrow \quad \text{"}\ (\bar{e}_2) = \langle x^2-1 \rangle$

$\Rightarrow \quad m_T(x) = (x-1)(x+1)(x-2)$

$\deg \mathrm{char}_T = 3 = \dim(V)$

$\Rightarrow m_T(x) = \mathrm{char}_T \quad \Rightarrow \quad V = \mathbb{R}[x]/(x-1)(x+1)(x-2)$ ← $\det(x \cdot I - T)$

$(x^2-1)(x-2) = x^3 - 2x^2 - x + 2$

w.r.t basis $1, x, x^2$ $\qquad\qquad x^3 = 2x^2 + x - 2$

$$[T] = \begin{array}{c|ccc} & 1 & x & x^2 \\ \hline 1 & 0 & 0 & -2 \\ x & 1 & 0 & 1 \\ x^2 & 0 & 1 & 2 \end{array}$$

If we decompose by elementary divisors (prime factor decomp).

$$V \cong k[x]/(p_1(x))^{a_1} \oplus \cdots \oplus$$

$a_1 + \cdots +$ is a partition of the power of p_1 dividing char poly.

In $k[x]/(p_1(x))^{a_1}$, $[T]$ has a nice form.

Let's focus on the case where p_1 is linear.

$p_1(x) = x - b$, then our basis is

$1,\ x-b,\ (x-b)^2,\ \ldots$

$$\begin{array}{c|ccc} & 1 & x-b & (x-b)^2 \\ \hline 1 & b & & \\ x-b & 1 & b & \\ (x-b)^2 & & 1 & \ddots \end{array}$$

$$[T] = \begin{bmatrix} b & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & b \end{bmatrix}$$