Lecture 22 - Structure Theory

Note Title 4/10/2008

Thm If M is a f.g. torsion module, and

$$M \cong Rw_1 \oplus \dots \oplus Rw_k \qquad \mathrm{Ann}(w_1) \supseteq \mathrm{Ann}(w_2) \supseteq \dots \supseteq \mathrm{Ann}(w_k)$$
$$\cong Rx_1 \oplus \dots \oplus Rx_\ell \qquad \text{" \quad " \quad "}$$

then $k = \ell$, and $\mathrm{Ann}(w_i) = \mathrm{Ann}(x_i)$ for all i.

Pf (cont): Saw $k=\ell$ by choosing a prime ideal $\langle p\rangle \supseteq \mathrm{Ann}(w_1)$ and reducing mod p.

$$M/p \cong Rw_1/pRw_1 \oplus \dots \oplus Rw_k/pRw_k = (R/p)^k$$
$$\cong Rx_1/pRx_1 \oplus \dots \oplus Rx_k/pRx_k = (R/p)^k$$

$$\Rightarrow Rx_i \supsetneq pRx_i \Rightarrow \langle p\rangle \supseteq \mathrm{Ann}(x_i)$$

If $\mathrm{Ann}(x_i) + \langle p\rangle = R \longleftrightarrow 1 = \underset{\mathrm{Ann}(x_i)}{\underset{\uparrow}{a}} + q\cdot \underset{\langle p\rangle}{\underset{\uparrow}{p}}$

$$\rightarrow x_i = 1\cdot x_i = (a+qp)(x_i)$$
$$= ax_i + qpx_i = qpx_i \quad \text{for all } x_i$$
$$\Rightarrow x_i \in pRx_i \Rightarrow Rx_i/pRx_i = \{0\}. \rightarrow\leftarrow$$

By induction on l = # of factors of a gen of $\mathrm{Ann}(M) = \langle a\rangle$.

If $\langle a\rangle \subseteq \langle b\rangle$, then b divides a.

$l=1$, then $\mathrm{Ann}(M) = \langle p\rangle$, so M is an R/p - vector space, so done.

$$\left(\mathrm{Ann}(M) = \mathrm{Ann}(w_k) = \mathrm{Ann}(x_k) \subseteq \dots \subseteq \underset{\mathrm{Ann}(x_1)}{\mathrm{Ann}(w_1)} \right)$$

Consider p a prime dividing a, and look at

$$pM \cong pRw_1 \oplus \dots \oplus pRw_k = pRw_s \oplus \dots \oplus pRw_k$$
$$(\mathrm{Ann}(w_1) = \dots = \mathrm{Ann}(w_{s-1}) = \langle p\rangle)$$
$$\cong pRx_1 \oplus \dots \oplus pRx_k = pRx_t \oplus \dots \oplus pRx_k$$

$\mathrm{Ann}(M) = \langle a\rangle \Rightarrow \mathrm{Ann}(pM) = \langle a/p\rangle$

$$\Rightarrow l(pM) < l(M), \text{ so by induction hyp,}$$

$s = t$, & $\mathrm{Ann}(pw_i) = \mathrm{Ann}(px_i)$ for all i.

If $\mathrm{Ann}(w_i) = \langle b_i \rangle$, $\mathrm{Ann}(x_i) = \langle c_i \rangle$, then

$$\mathrm{Ann}(pw_i) = \langle \tfrac{b_i}{p} \rangle = \mathrm{Ann}(px_i) = \langle \tfrac{c_i}{p} \rangle \Rightarrow \underbrace{\langle b_i \rangle}_{\mathrm{Ann}(w_i)} = \underbrace{\langle c_i \rangle}_{\mathrm{Ann}(x_i)}. \quad \boxtimes$$

<u>Def</u> If p a prime in R, then let $M_p = \{ m \in M \mid p^k \cdot m = 0, \text{ some } k \}$

(p-typic piece, p-primary piece, etc)

<u>Cor</u> $M \cong M_{p_1} \oplus \dots \oplus M_{p_k}$, p_i all divide a gen of $\mathrm{Ann}(M)$.

<u>Pf</u>: by previous thm, it suffices to show this for M cyclic.

$$R/(p_1^{a_1} \cdots p_k^{a_k}) \underset{\uparrow}{\cong} (R/p_1^{a_1}) \oplus \dots \oplus (R/p_k^{a_k})$$

Chinese Remainder Theorem. $\boxtimes$

F.G Abelian Groups

Thm says:

"given A, an abelian group, have a sequence $s_1 | s_2 | \dots$ of integers s.t. $A \cong \mathbb{Z}/s_1 \oplus \mathbb{Z}/s_2 \oplus \dots \oplus \mathbb{Z}/s_n$".

$\Rightarrow$ have a unique seq of positive (non-neg) ints $1 \neq s_1 | s_2 | \dots$ s.t.

$$A \cong \mathbb{Z}/s_1 \oplus \dots \oplus \mathbb{Z}/s_n .$$

$|A| = s_1 \cdots s_n$, s_n gn the $\mathrm{Ann}(A)$.

If $|A| = p_1 \cdot p_2 \cdot \dots \cdot p_k \Rightarrow A = \mathbb{Z}/(p_1 \cdot p_2 \cdots p_k)$

If $|A| = 25$ $\quad 25 \leftarrow s_1 \rightsquigarrow A = \mathbb{Z}/25$

$5 \leftarrow s_1 = s_2 \rightsquigarrow A = \mathbb{Z}/5 \times \mathbb{Z}/5$

$\times$

If $|A| = p_1^2 p_2 \cdots p_k \Rightarrow A = \mathbb{Z}/(p_1^2 \cdots p_k)$

$s_1 = p_1$, $s_2 = p_1 \cdots p_k \Rightarrow \mathbb{Z}/p_1 \times \mathbb{Z}/p_1 \cdots p_k$

Any abelian group A can be written

$$A = \left(\mathbb{Z}/p_1^{e_1} \oplus \cdots \oplus \mathbb{Z}/p_1^{e_k}\right) \oplus \cdots \oplus \left(\mathbb{Z}/p_n^{f_1} \oplus \cdots \oplus \mathbb{Z}/p_n^{f_\ell}\right)$$

where $|A| = \left(p_1^{e_1+\cdots+e_k}\right) \cdots \left(p_n^{f_1+\cdots+f_\ell}\right)$

Structure theory shows: given k, any partition of k gives a distinct abelian group of order $\mathbb{Z}/p^k$:

$$k = n_1 + n_2 + \cdots + n_\ell \rightsquigarrow s_1 = p^{n_1}, \ s_2 = p^{n_2}, \ \ldots, \ s_\ell = p^{n_\ell}$$

Ex: $54 = 27 \cdot 2 = 3^3 \cdot 2$

$s_1 \mid s_2 \mid s_3 \mid s_4$

54

$3 \mid 18$

$3 \mid 3 \mid 6$

		2	2 primary
		3^3	
	3	3^2	3 primary
3	3	3	