Lecture 21 - Structure Theory

Note Title 4/8/2008

<u>Thm</u> If M is a free module of rank $k<\infty$, and N is a sub mod. then there is a basis $\{m_1,\dots,m_k\}$ of M and $s_1,\dots,s_n\in R$ s.t.

1) $\{s_1m_1,\dots,s_km_k\}$ is a basis of N
2) $s_i \mid s_{i+1}$

<u>Pf</u> By induction on k.

$k=1$: $M=R$, $N\subseteq R \Rightarrow \exists\, a$ s.t. $N=aR$.

$k>1$: Consider for each $y\in N$ the ideal $\langle c(y)\rangle$.

The set $\{\langle c(y)\rangle \mid y\in N\}$ is a poset under $\subseteq$, since R Noetherian

$\Rightarrow \exists$ a maximal element. Choose y s.t. $\langle c(y)\rangle$ is maximal, and

let x_1 be s.t. $y=c(y)\cdot x_1$, so $s_1=c(y)$.

$\Rightarrow$ Can extend to a basis: $\{x_1, x_2',\dots x_k'\}$ of M.

<u>Claim</u> $N=\langle s_1x_1\rangle \oplus (\langle x_2',\dots,x_k'\rangle \cap N)$.

Let $w\in N$, then since $\{x_1,x_2',\dots,x_k'\}$ is a basis of M,

$$w=ux_1+\sum_i b_ix_i'$$

Let d be the $\gcd(u,s_1) \rightarrow$ can find a,b s.t. $au+bs_1=d$

$\Rightarrow\ z=aw+b(s_1x_1)\in N$

$= dx_1+\sum_i ab_ix_i'$

$c(z)$ divides $d \Rightarrow \langle s_1\rangle \subseteq \langle d\rangle \subseteq \langle c(z)\rangle$

maximality of $\langle s_1\rangle \Rightarrow \langle s_1\rangle = \langle c(z)\rangle \Rightarrow \langle s_1\rangle=\langle d\rangle$

$\Rightarrow s_1$ is the $\gcd(s_1,u) \Rightarrow s_1$ divides u.

$\Rightarrow ux_1\in\langle s_1x_1\rangle$

$\Rightarrow w\in N \Rightarrow w=\underbrace{\alpha\cdot(s_1x_1)}_{\in\,\langle s_1x_1\rangle}+\underbrace{\sum_i b_ix_i'}_{\in\,\langle x_2',\dots,x_k'\rangle\cap N}$

$\langle s_1x_1\rangle \cap (\langle x_2',\dots,x_k'\rangle\cap N)=\{0\}$ because $\{x_1,x_2',\dots,x_k'\}$ is a basis for M.

Let $N_1 = \langle x_2', \dots, x_k' \rangle \cap N$

Induction hyp $\Rightarrow$ $\langle x_2', \dots, x_k' \rangle$ has a basis $\{x_2, \dots, x_k\}$ & have #s $s_2, \dots, s_k \in R$ s.t. $\{s_2x_2, \dots, s_kx_k\}$ is a basis for N_1 & $s_i \mid s_{i+1}$.

$\Rightarrow$ $\{x_1, \dots, x_k\}$ is a basis for M. & $s_1, \dots, s_k$ are s.t.

$\{s_1x_1, s_2x_2, \dots, s_kx_k\}$ is a basis for N,

& $s_i \mid s_{i+1}$ $i \geq 2$

$s_1 \mid s_2$: $z = s_1x_1 + s_2x_2 \in N$

let $d = \gcd(s_1, s_2)$, then $d = c(z)$ &

$\langle s_1 \rangle \subseteq \langle d \rangle = \langle c(z) \rangle$

$\uparrow$ maximal $\Rightarrow$ $\langle s_1 \rangle = \langle c(z) \rangle$ $\Rightarrow$ s_1 divides d $\Rightarrow$

$s_1 \mid s_2$. ☒

Cor If M is f.g. then

$$M \cong (R \oplus \dots \oplus R) \oplus R/(s_1) \oplus R/(s_2) \oplus \dots \oplus R/(s_k) \quad \text{some}$$

$$s_1 \mid s_2 \mid s_3 \mid \dots \mid s_k.$$

Pf: M is f.g, so there is a surjective map

$$R^n \xrightarrow{\phi} M \quad \text{some } n.$$

$\ker \phi$ is a submodule of a f.g. free mod, so by previous thm, there is a basis $\{x_1, \dots, x_n\}$ of R^n and numbers $s_1 \mid s_2 \mid \dots \mid s_k \mid s_{k+1} \mid \dots \mid s_n$ s.t.

$\{s_1x_1, \dots, s_nx_n\}$ is a basis for $\ker \phi$.

$$N \oplus M / N_1 \oplus M_1 \cong N/N_1 \oplus M/M_1$$

1^{st} Isom thm: ϕ induces an isom

$$R^n / \ker \phi \xrightarrow{\tilde{\phi}} M$$

$$\Uparrow$$

$$N + M / N \cong N / N \cap M$$

if $N \cap M$ is trivial, then

$$N \oplus M / N \cong M$$

By 2^{nd} Isom theorem:

$$\begin{array}{ccccccccc} R^n & = & Rx_1 & \oplus & Rx_2 & \oplus & \cdots & \oplus & Rx_n \\ \cup\!\!\mid & & \cup\!\!\mid & & \cup\!\!\mid & & & & \cup\!\!\mid \\ \ker\phi & & Rs_1x_1 & \oplus & Rs_2x_2 & \oplus & \cdots & \oplus & Rs_nx_n \end{array}$$

$$R^n / \ker \phi \cong \left(Rx_1 / Rs_1x_1\right) \oplus \left(Rx_2 / Rs_2x_2\right) \oplus \dots \oplus \left(Rx_n / Rs_nx_n\right)$$

$$\wr\!\!\mid\!\!\mid \qquad\qquad \wr\!\!\mid$$

$$\left(R/(s_1)\right) \oplus R/(s_2) \oplus \dots \oplus \left(R/(s_n)\right)$$

☒

$$N \oplus M \big/ N_1 \oplus M_1 \cong N/N_1 \oplus M/M_1$$

$$\begin{aligned} N \oplus M &\longrightarrow N/N_1 \oplus M/M_1 \qquad \text{is surjective} \\ (n,m) &\longmapsto ([n],[m]) \end{aligned}$$

If $(n,m) \mapsto (0,0) = ([n],[m])$, then $[n]=[0]=N_1 \longleftrightarrow n \in N_1$

$[m]=[0]=M_1 \longleftrightarrow m \in M_1$

$\longleftrightarrow (n,m) \in N_1 \oplus M_1$.

We don't exclude the case $s_j = 0$: $R/s_j = R/0 = R$

$\Rightarrow$ gives free factors in our direct sum decomp.

If $s_j \neq 0$, then R/s_j is torsion, & $R/s_1 \oplus \cdots \oplus R/s_k = M_\tau$

<u>Prop</u> If M is f.g, then $M \cong M_\tau \oplus F$, F free.

If $f: M \to N$ is a hom, then f is an isom iff

$f|_{M_\tau} : M_\tau \xrightarrow{\cong} N_\tau$ and $\tilde{f} : M/M_\tau \xrightarrow{\cong} N/N_\tau$.

restriction $\qquad\qquad$ universal property of quotients.

<u>Pf</u>:

$m \longmapsto \pi(m)$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & M_\tau \arrow[r] \arrow[d, "f|_{M_\tau}"] & M \arrow[r] \arrow[d, "f"] & F_M \arrow[r] \arrow[d, "\tilde{f}"] & 0 \arrow[d, equal] \\ 0 \arrow[r] & N_\tau \arrow[r] & N \arrow[r] & F_N \arrow[r] & 0 \end{tikzcd}$$

$f|_{M_\tau}$ & $\tilde{f}$ are iso. $\Rightarrow$ f is an isom.

f is inj: $\qquad f(m) = 0$

Consider $\pi(m)$ & $\tilde{f}(\pi(m)) = \pi(f(m)) = 0$

$\tilde{f}$ injective $\Rightarrow \pi(m) = 0 \Rightarrow m \in M_\tau$

$f|_{M_\tau}$ is an iso, so $\underset{\overset{\|}{0}}{f(m)} = f|_{M_\tau}(m) \Rightarrow m = 0$

Surjectivity is similar. $\boxtimes$

To show our decomp is unique, then it suffices to show uniqueness for torsion modules.

Thm If M is a f.g. torsion module, and

$$M \cong Rw_1 \oplus \cdots \oplus Rw_k \qquad \mathrm{Ann}(w_1) \supseteq \mathrm{Ann}(w_2) \supseteq \cdots \supseteq \mathrm{Ann}(w_k)$$

$$\cong Rx_1 \oplus \cdots \oplus Rx_\ell \qquad \text{"} \qquad \text{"} \qquad \text{"}$$

then $k = \ell$, and $\mathrm{Ann}(w_i) = \mathrm{Ann}(x_i)$ for all i.

$$R/(s_i) = Rw_i \quad \leftrightarrow \quad \mathrm{Ann}(w_i) = \langle s_i \rangle$$

$$s_i \mid s_{i+1} \quad \longleftrightarrow \quad \langle s_i \rangle \supseteq \langle s_{i+1} \rangle$$

$$\Updownarrow$$

$$\mathrm{Ann}(w_i) \supseteq \mathrm{Ann}(w_{i+1}).$$

<u>Pf</u>: $\mathrm{Ann}(M) = \mathrm{Ann}(w_k) = \mathrm{Ann}(x_\ell) \neq \{0\}$.

Choose p a prime dividing s_1 $\quad (\langle p \rangle \supseteq \mathrm{Ann}(w_1))$

Then R a PID, $\langle p \rangle$ is maximal, so R/p is a field $\Rightarrow$

M/p is an R/p-vector space

$$M/p \cong (Rw_1)/(Rp\,w_1) \oplus \cdots \oplus (Rw_k)/(pRw_k).$$

for all i, $\quad Rw_i/pRw_i \neq \{0\}$. $\quad$ ie $Rw_i \neq pRw_i$

Assume $\quad w_i = a \cdot p \cdot w_i$ $\quad$ some a $\quad$ (ie $Rw_i \subseteq pRw_i$)

$$\Rightarrow (1 - ap) w_i = 0 \quad \Rightarrow \quad 1 - a \cdot p \in \mathrm{Ann}(w_i) \subseteq \langle p \rangle.$$

$$\Rightarrow 1 \in \langle p \rangle, \text{ contradicting } p \text{ prime.}$$

$$\Rightarrow M/p \cong \underbrace{R/p \oplus \cdots \oplus R/p}_{k}$$

$$M/p \cong (Rx_1/pRx_1) \oplus \cdots \oplus (Rx_\ell/pRx_\ell) \quad \Rightarrow \ell \geq k$$