Lecture 20 – Free Modules over a PID (II)

Note Title 4/3/2008

Let S be a mult. subset of R ($0 \notin S$). Then we can form

$$R \to R_S = R \times S/\sim$$
$$= \left\{ \frac{r}{s} \right\}$$

Given an R-mod M, let $\sim$ on $M \times S$ be $(m,s) \sim (n,t)$ iff $tm = sn$

Def: The localization of M is $M \times S/\sim$.

$$(m,s) \longleftrightarrow \frac{m}{s}.$$

Prop 1) M_S is an R-mod via $r \cdot \frac{m}{s} = \frac{rm}{s}$ &

$$\frac{m}{s} + \frac{n}{t} = \frac{tm+sn}{st}$$

2) M_S is an R_S-mod via $\frac{r}{s} \cdot \frac{m}{t} = \frac{rm}{st}$.

Pf: Homework $\boxtimes$

Prop: - $M \longmapsto M_S$ is a functor from R-mod to R_S-mod.

- It is an exact functor.

Pf: $(f: N \to M) \longmapsto (f_S : N_S \to M_S)$

$$f_S\left(\frac{n}{t}\right) = \frac{f(n)}{t}$$ $\boxtimes$

Remark: $M_S = R_S \otimes_R M$. "Flat base change"

If R is a PID, & $S = R - \{0\}$, then $R_S = Q(R)$ is a field.

$(-)_S$ is a functor $R\text{-mod} \longrightarrow Q(R)\text{-mod} = Q(R)$ vectorspaces.

Thm If M is a free R-mod, then any two bases have the same # of elements.

Pf: Choose 2 bases: X, Y of M

$$X \longleftrightarrow \bigoplus_{x \in X} R \xrightarrow{\sim} M$$

$$Y \longleftrightarrow \bigoplus_{y \in Y} R \xrightarrow{\sim} M$$

Apply $(-)_S$: $\left(\bigoplus_{x\in X} R\right)_S \cong M_S \cong \left(\bigoplus_{y\in Y} R\right)_S$

$\left(\bigoplus_{x\in X} R\right)_S = \bigoplus_{x\in X} Q(R)$, $\quad \left(\bigoplus_{y\in Y} R\right)_S = \bigoplus_{y\in Y} Q(R)$

$\Rightarrow \dim_{Q(R)} M_S = |Y|$

$\dim_{Q(R)} M_S = |X| = \dim_{Q(R)} \bigoplus_{x\in X} Q(R)$ $\quad\boxtimes$

Def The rank of a free module M is the cardinality of any basis.

Def An element x of a module M is primitive / indecomposable.

if wherever $x = a\cdot x'$, $a\in R$, $x'\in M$, a is a unit.

Prop If M is f.g, then for any $x\in M$, $\exists\ c(x)\in R$ s.t.

$x = c(x)\cdot x'$, $\quad x'$ indecomposable.

Pf: x not indecomp $\Rightarrow$ $x = a\cdot x_1$, $\quad a\notin R^\times$.

$\Rightarrow \langle x\rangle \subseteq \langle x_1\rangle$

Repeat w/ x_1

$\Rightarrow$ get a chain $\quad \langle x\rangle \subseteq \langle x_1\rangle \subseteq \dots$

Since R is Noetherian, this chain stabilizes. $\Rightarrow \exists\, n$ s.t. $\langle x_i\rangle = \langle x_{i+1}\rangle\ \forall i\geq n$.

x_n is our x'.

Def $c(x)$ is the content / carrier of x.

Thm If M is f.g & free, then if x is indecomp, then there is a basis of M containing x.

Pf By induction on rank $M = k$

$k=1$) obvious $\{x\}$ is a basis.

$k=2$) $\{x_1, x_2\}$ be a basis for M. Write $x = rx_1 + sx_2$.

Then $\gcd(r,s) = 1$ $\Rightarrow$ can find a, b s.t. $ar + bs = 1$

$\begin{bmatrix} r & -b \\ s & a \end{bmatrix}$ is invertible in $M_2(R)$.

$\Rightarrow \{x = rx_1 + sx_2,\ x' = -bx_1 + ax_2\}$ is a basis.

$k=n$) $\{x_1, \dots, x_n\}$ a basis. Write $x = \underbrace{\sum a_i x_i}_{y} + a_n x_n$

1) $a_n = 0 \Rightarrow x \in \mathrm{Span}(x_1, \ldots, x_{n-1}) \Rightarrow$ induction $\Rightarrow \{x, x_2', \ldots, x_{n-1}', x_n\}$

2) $y = 0 \Rightarrow x \in \mathrm{Span}(x_n) \Rightarrow$ ind (replace x_n w/ x)

$y \neq 0$, let y' be the indecomp element w/

$y = b \cdot y' \Rightarrow$ extend to a basis for $\mathrm{span}(x_1, \ldots, x_{n-1})$

$x = b \cdot y' + a_n x_n \Rightarrow \gcd(b, a_n) = 1 \Rightarrow (k=2) \Rightarrow$ a basis

$\{x, ?y' + ?x_n, y_2', \ldots, y_{n-1}'\}$. $\boxtimes$

Thm If M is a free module of rank $k < \infty$, and N is a sub mod. then there is a basis $\{m_1, \ldots, m_k\}$ of M and $s_1, \ldots, s_n \in R$ s.t.

1) $\{s_1 m_1, \ldots, s_k m_k\}$ is a basis of N

2) $s_i \mid s_{i+1}$

Pf By induction on k.

$k=1$: $M = R$, $N \subseteq R \Rightarrow \exists a$ s.t. $N = aR$.

$k>1$: Consider for each $y \in N$ the ideal $\langle c(y) \rangle$.

The set $\{\langle c(y) \rangle \mid y \in N\}$ is a poset under $\subseteq$, since R Notherian

$\Rightarrow \exists$ a maximal element. Choose y s.t. $\langle c(y) \rangle$ is maximal, and let x_1 be s.t. $y = c(y) \cdot x_1$, so $s_1 = c(y)$.

$\Rightarrow$ Can extend to a basis: $\{x_1, x_2', \ldots x_n'\}$ of M.

Claim $N = \langle s_1 x_1 \rangle \oplus (\langle x_2', \ldots, x_n' \rangle \cap N)$.