Lecture 19 - Free Modules over a P.I.D.

Note Title 4/1/2008

Thm If R is a PID, then any submodule of a free module is free (can ensure the basis has at most the number of elements in the basis for bigger module).

Def A partially ordered set is *well-ordered* if every subset has a least element: $Y \subseteq X$, $\exists\, y \in Y \mid y \leq y'$, $y' \in Y$

Thm Every set admits a well-ordering.

If $y \in X$, X well ordered, the *successor to y*, $y+1$, is the least element in $\{z \mid y < z\}$.

Pf of Thm

Choose a basis $\{x_i\}_{i \in I}$ for M, our free module.

Pick a well ordering of I: $\leq$ & define

$M_i = \langle x_j \mid j < i \rangle$

(1) If $i < j$, then $M_i \subseteq M_j$ $\quad$ ($x_i \in M_j$ since $i<j$. $x_i \notin M_i$, $i \not< i$

(2) $M = \bigcup_{i \in I} M_i$

(3) $M_{i+1}/M_i \cong R \cdot x_i$ $\qquad$ $\left(\{k \mid k < i\} \subseteq \{k \mid k < i+1\} = \{k \mid k < i\} \cup \{i\} \right)$

Let $N \subseteq M$ be a submodule,

$N_i = N \cap M_i$

(4) $N = \bigcup_{i \in I} N_i$

(5) If $i<j$, $N_i \subseteq N_j$

(6) $N_i = N_{i+1} \cap M_i = (N \cap M_{i+1}) \cap M_i = N \cap (M_{i+1} \cap M_i) = N \cap M_i$

$$N_{i+1}/N_i = N_{i+1}/N_{i+1} \cap M_i \underset{\text{2nd Iso Thm}}{\cong} N_{i+1} + M_i/M_i \subseteq M_{i+1}/M_i \cong R$$

$\Rightarrow$ N_{i+1}/N_i is a submodule of R.

$\Leftrightarrow$ an ideal $\underset{\text{PID}}{\longleftrightarrow}$ a copy of R.

$a \cdot R \longleftarrow R$

$$N_{i+1}/N_i \xrightarrow{\sim} a \cdot R \subseteq R$$

some a

If $a \neq 0$: $R \xrightarrow{\sim} a \cdot R$, $x \longmapsto ax$ (ob. onto $b \in \ker$, $ab = 0 \Rightarrow b = 0$)

If $a = 0$: $N_{i+1}/N_i = \{0\} \longleftrightarrow N_{i+1} = N_i$

$\Rightarrow$ $0 \to N_i \to N_{i+1} \to a \cdot R \to 0$

$\Rightarrow$ $N_{i+1} = N_i \oplus R \cdot b$ some $b \in N_{i+1}$.

Let b_{i+1} be this b.

$\{b_i\}_{i \in I}$ is basis for N.

Spans & is lin ind.

"by induction" : If A well ordered & B is a subset s.t.

$$\underbrace{\{a \in B \ \forall a < c\}}_{\{a < c\} \subseteq B} \Rightarrow c \in B$$

$\Rightarrow B = A$.

Spans: $n \in N = \bigcup_{i \in I} N_i \Rightarrow n \in N_i$ some i. Write $i = j+1$

$N_i = N_j \oplus R b_{j+1}$, $n = n_j + r_i b_{j+1}$, $n_j \in N_j$

by the induction hypothesis, $n_j \in \operatorname{Span}(b_k, k \leq j)$

$\Rightarrow n \in \operatorname{Span}(b_k, k \leq j+1)$

Lin Ind is the same. ☒

$\mathbb{Q}$ as a $\mathbb{Z}$-module, admits a filtration by $\mathbb{Z}$s.

$$\mathbb{Z} \subseteq \mathbb{Z}\{\tfrac{1}{2}\} \subseteq \mathbb{Z}\{\tfrac{1}{6}\} \subseteq \mathbb{Z}\{\tfrac{1}{24}\} \subseteq \mathbb{Z}\{\tfrac{1}{120}\} \subseteq \dots \subseteq \mathbb{Z}\{\tfrac{1}{n!}\} \subseteq \dots$$

$$\bigcup \mathbb{Z}\{1/n!\} = \mathbb{Q}.$$

Cor: Any module has a resolution of length 2 by frees:

Pf

$$0 \to K \to F \to M \to 0$$

(free $\Leftarrow$ free) ☒

Cor: Projectives are free.

Pf: P proj $\Rightarrow$ P summand of free $\Rightarrow$ P submod of free $\Rightarrow$ P free ☒

Thm If M is f.g. and torsion free, then M is free.

Pf Let X be a (minimal) spanning set, and choose $S \subseteq X$ maximal w.r.t. $\subseteq$ s.t. S is lin. ind.

If $x \in X$, then $\{x\} \cup S$ is lin dep $\Rightarrow$ can find

$0 \neq r_x, r_1, \dots, r_k$ s.t. $r_x \cdot x + \underbrace{r_1 s_1 + \dots + r_k s_k}$ $= 0$ $\quad s_i \in S$.

$\Rightarrow r_x x \subseteq \mathrm{Span}(S)$

Since X is finite, $\exists\, r \in R$ s.t. $rx \in \mathrm{Span}(S)$ $\forall\, x \in X$

$\left(r = \prod_{y \in X - S} r_y\right)$.

$\Rightarrow \underset{\substack{\| \\ r \cdot M}}{\mathrm{Span}(r \cdot X)} \subseteq \mathrm{Span}(S) \leftarrow$ free

$\Rightarrow r \cdot M$ is a submodule of a free module $\Rightarrow r \cdot M$ is free.

Have a map $M \xrightarrow{L_r} rM$

$m \longmapsto rm$

$\ker(L_r) = \{m \mid r \cdot m = 0\} = \{0\} \Rightarrow L_r$ is an isomorphism

$\Rightarrow M$ is free. $\blacksquare$