Lecture 18 - Projectives & Frees

Note Title 3/27/2008

<u>Ex</u> •) $0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n \to 0$

$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, -)$:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[dr] & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \arrow[r, "(\cdot n)_*"] \arrow[d, equal] & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/n) \arrow[d, equal] \\
& 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/n
\end{tikzcd}
$$

•) $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

$\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \arrow[d, equal] \\
& 0 & 0 & \mathbb{Z}
\end{tikzcd}
$$

<u>Remark</u>: There is a functor that measures the failure of right exactness: $\operatorname{Ext}(-,-)$

for $\mathbb{Z}$-modules:

$$0 \to A \to B \to C \to 0$$

$$0 \to \operatorname{Hom}(D,A) \to \operatorname{Hom}(D,B) \to \operatorname{Hom}(D,C) \to \operatorname{Ext}(D,A) \to$$
$$\operatorname{Ext}(D,B) \to \operatorname{Ext}(D,C) \to 0$$

$\operatorname{Hom}(-,-)$ is exact on split exact sequences.

$$
\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "f"] & M \arrow[r, shift left, "g"] & M_2 \arrow[l, dashed, shift left, "\beta"] \arrow[r] & 0
\end{tikzcd}
, \quad \text{split, then}
$$

$$
\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(N,M_1) \arrow[r] & \operatorname{Hom}(N,M) \arrow[r, shift left, "g_*"] & \operatorname{Hom}(N,M_2) \arrow[l, dashed, shift left, "\beta_*"] \arrow[r] & 0
\end{tikzcd}
\quad \text{is split exact}
$$

Given $h \in \operatorname{Hom}(N, M_2)$, $\beta_*(h) \in \operatorname{Hom}(N,M)$ is a preimage under g_*

ie $g_*(\beta_*(h)) = h.$

$$g \circ (\beta \circ h) = (g \circ \beta) \circ h = Id_{M_2} \circ h$$

$$g_* \circ \beta_* = Id_{\operatorname{Hom}(N,M_2)}.$$

$$\operatorname{Hom}(M \oplus N, P) = \operatorname{Hom}(M,P) \oplus \operatorname{Hom}(N,P)$$

$$\mathrm{Hom}(P, M\oplus N) = \mathrm{Hom}(P,M)\oplus \mathrm{Hom}(P,N)$$

Def · An R-mod P is *projective* if $\mathrm{Hom}(P,-)$ is right exact.

· An R-mod I is *injective* if $\mathrm{Hom}(-,I)$ is right exact.

Remark: $\mathrm{Ext}(P,-)=0$, $\mathrm{Ext}(-,I)=0$

Ex: If M is a free R-mod, then M is projective.

$\mathrm{Hom}(P,-)$ is right exact $\longleftrightarrow$ given any surjection $M \xrightarrow{f} N$ & a map $P\to N$, we have a lift $P\to M$ s.t.

$$\begin{tikzcd} P \arrow[r] \arrow[dr, dashed] & N \\ & M \arrow[u] \end{tikzcd} \qquad \begin{array}{c} \in \mathrm{Hom}(P,N) \\ \uparrow f_* \\ \in \mathrm{Hom}(P,M) \end{array}$$

Prop The free R-module functor is adjoint to the forgetful functor.

$$\mathcal{S}et \xrightarrow{F} R\text{-mod} \qquad\qquad R\text{-mod} \xrightarrow{\mathcal{U}} \mathcal{S}et$$

$$X \longrightarrow \underset{X}{R\text{-mod w/ basis}} \qquad\qquad (M,+,\cdot) \longmapsto M$$

$$\mathrm{Hom}_R(F(X), M) = \mathrm{Hom}_{\mathcal{S}et}(X, \mathcal{U}(M)).$$

Cor Frees are projective

$$\begin{tikzcd} F(X) \arrow[r, "f"] \arrow[dr, dashed] & N \\ & M \arrow[u, "\text{surjective}"'] \end{tikzcd} \longleftrightarrow \quad X \xrightarrow{f} \mathcal{U}(N) \qquad n_x = f(x)$$

so $\forall x$, can find $m_x \longmapsto n_x$

$\Longrightarrow$ a function

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & \mathcal{U}(M) \arrow[d] \\ & \mathcal{U}(N) \end{tikzcd}$$

Prop: If P projective, then P is a summand of a free module.

$\longleftrightarrow$ $\exists P'$ s.t. $P\oplus P' = $ free.

Pf: Choose generators of P $X = \{x_i\}_{i\in I}$.

There is a surjective map $F(X)\to P$ $\Rightarrow$ a S.E.S

$$0\to K\to F(X)\xrightarrow{\pi} P\to 0$$

Look at $\mathrm{Hom}(P,-)$:

$$0\to \mathrm{Hom}(P,K)\to \mathrm{Hom}(P,F(X))\xrightarrow{\pi_*}\mathrm{Hom}(P,P)\to 0$$

$$(f: P \to F(x)) \longmapsto 1_P$$

ie: $\exists\ f: P \to F(x)$

s.t. $\pi \circ f = Id. \Rightarrow f$ is our splitting.

$\Rightarrow F(x) = P \oplus K$ $\boxtimes$

Cor: $\{P_i\}_{i \in I}$ is a family of proj modules iff $\bigoplus_{i \in I} P_i$ is projective.

Pf P_i all proj $\Rightarrow \exists\ P_i'$ s.t. $P_i \oplus P_i' = F(x_i)$.

$$\bigoplus_{i \in I} P_i \oplus \bigoplus_{i \in I} P_i' = \bigoplus_{i \in I} P_i \oplus P_i' = \bigoplus_{i \in I} F(x_i) = F(\cup x_i)$$

$\bigoplus P_i$ is proj $\Rightarrow \exists P'$ s.t. $(\bigoplus P_i) \oplus P' = F(X)$

$$= (P_i \oplus \bigoplus_{j \neq i} P_j) \oplus P'$$

$$= P_i \oplus (\bigoplus_{i \neq j} P_j \oplus P')$$ $\boxtimes$

Ex: If R is a PID, then projective $\iff$ free. (Next time)

Ex: $R = \mathbb{Z}/2 \times \mathbb{Z}/3 = \mathbb{Z}/6$ $(\mathbb{Z}/n \times \mathbb{Z}/m)$

$P = \mathbb{Z}/2$ $\quad Q = \mathbb{Z}/3$ & as R-mod, $P \oplus Q = R$

$(a,b) \cdot m = am$

"Remark"

Def A module D is divisible if $\forall\ a \in D,\ n \in \mathbb{N},\ \exists\ b$ s.t. $n.b = a$.

Ex: $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{Z}/p^\infty = \mathbb{Z}[1/p]/\mathbb{Z}$

Thm: Injective $\leftrightarrow$ divisible.

"Remark"

Computing Ext(,): $\quad$ Ext(A, B)

1) form a projective / free resolution of A:

$$\cdots \to P_2 \xrightarrow{f_1} P_1 \xrightarrow{f_0} A \to 0$$

projective.

Apply Hom($\cdot$, B)

$$0 \to \mathrm{Hom}(A, B) \xrightarrow{f_0^*} \mathrm{Hom}(P_1, B) \xrightarrow{f_1^*} \cdots$$

$$\mathrm{Ext}^i(A,B) = \ker(f_{i+1}^*)/\mathrm{Im}(f_i^*)$$

$\mathbb{Z}/n$:
$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

$\mathrm{Ext}^i(\mathbb{Z}/n, \mathbb{Z})$:
$$0 \to \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}) \to \underset{\mathbb{Z}}{\underset{\shortparallel}{\mathrm{Hom}(\mathbb{Z},\mathbb{Z})}} \xrightarrow{\cdot n} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \to 0$$

$$\mathbb{Z} \to 0$$

$$\Rightarrow \quad \mathbb{Z}/n = \mathrm{Ext}^1(\mathbb{Z}/n, \mathbb{Z})$$