Lecture 17 – Torsion & Hom(-,-)

Note Title 3/25/2008

If M is an R-mod, $M_\tau = \{x \in M \mid Ann(x) \neq 0\}$

If R is an ID, then M_τ is a (torsion) submodule.

Ex: $R = \mathbb{F}_2[x,y]/\langle xy \rangle$

R is an R-module, $x, y \in R_\tau$

$y \cdot x = 0 \Rightarrow Ann(x) \ni y$

$x \cdot y = 0$

$Ann(x+y) = \{0\}$ (wants to be $xy = 0$)

If R is an ID, then M/M_τ is torsion free.

$a \cdot m$ is torsion $\Rightarrow$ m is torsion

$\Downarrow$

$\exists\ r$ s.t. $r \cdot (a \cdot m) = 0$

$\phantom{\exists\ r \text{ s.t. }} \parallel$

$(r \cdot a) \cdot m$

$\neq$

0 $\Rightarrow r \cdot a \in Ann(m)$

Consequence: Given any M, there is a canonical short exact sequence of R-modules:

$$0 \longrightarrow M_\tau \xrightarrow{f} M \xrightarrow{g} M/M_\tau \longrightarrow 0$$

where M_τ is torsion & M/M_τ is torsion free.

Injective surjective

$ker(g) = Im(f)$

Def: A sequence of R-modules

$$\cdots \longrightarrow M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \longrightarrow \cdots$$

- is **exact** if $ker(f_n) = Im(f_{n+1})$
- is a **chain complex** if $Im(f_{n+1}) \subseteq ker(f_n)$

Ex: In $\mathbb{Z}$-modules:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

1) $\mathrm{Im}(0 \to \mathbb{Z}) = \ker(\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z})$ general: $0 \to N \xrightarrow{f} M$ exact $\leftrightarrow$ f inj.

2) $\mathrm{Im}(\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}) = \ker(\mathbb{Z} \to \mathbb{Z}/n)$ ✓

3) $\mathrm{Im}(\mathbb{Z} \to \mathbb{Z}/n) = \ker(\mathbb{Z}/n \to 0)$ general: $M \xrightarrow{g} N' \to 0$ exact $\leftrightarrow$ g surj.

<u>Ex</u>: 1) $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

2) $0 \to M_\tau \to M \to M/M_\tau \to 0$

If $0 \to N \xrightarrow{f} M \xrightarrow{g} N' \to 0$, then g induces an iso
$$M/N \xrightarrow{\bar{g}} N'$$
$\ker(g) = \mathrm{Im}(f) \cong N$

3) If R is a field, then if $0 \to N \to M \to N' \to 0$ is exact, then $M \cong N \oplus N'$

<u>Def</u> A short exact seq is split if $M \cong N \oplus N'$

<u>Prop</u> $0 \to N \xrightarrow{f} M \xrightarrow{g} N' \to 0$ is split iff

1) $\exists$ $\alpha: M \to N$ s.t. $\alpha \circ f = Id_N$

2) $\exists$ $\beta: N' \to M$ s.t. $g \circ \beta = Id_{N'}$

<u>Pf</u>: split $\Rightarrow$ 1, 2 $M \cong N \oplus N'$: $\alpha = \pi_N$, $(n,n') \mapsto n$, $\beta = i_{N'}: N' \to M$, $n' \mapsto (0,n')$

1) $\Rightarrow$ split. $m \in M$

$\alpha(m) \in N$

$f(\alpha(m)) \in M$, in $\mathrm{Im}(f)$

$m - f(\alpha(m)) \in M$

$$\alpha(m - f(\alpha(m))) = \alpha(m) - \alpha(f(\alpha(m)))$$
$$= \alpha(m) - \alpha(m) = 0$$

$$m = \underset{\ker(\alpha)}{\underset{\cap}{(m - f(\alpha(m)))}} + \underset{\mathrm{Im}(f)}{\underset{\cap}{f(\alpha(m))}}$$

Assume $p \in \ker(\alpha) \cap \mathrm{Im}(f)$

$\Updownarrow$

$\alpha(p) = 0$ & $\exists\, n$ s.t. $p = f(n)$

$0 = \alpha(p) = \alpha(f(n)) = n \Rightarrow p = 0$ 1^{st} Iso Theorem.

$\Rightarrow M \cong \ker(\alpha) \oplus \mathrm{Im}(f) \cong \ker(\alpha) \oplus N \cong N' \oplus N$

<u>Recall</u> A functor from $\mathcal{C} \to \mathcal{D}$ is

1) $F : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$

2) $F : \mathrm{Hom}(a,b) \to \mathrm{Hom}(F(a), F(b))$

$(F(Id_A) = Id_{F(A)}, \quad F(f \circ g) = F(f) \circ F(g))$

= covariant

<u>Def</u> A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is

1) $F : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$

2) $F : \mathrm{Hom}(a,b) \to \mathrm{Hom}(F(b), F(a))$

$(F(Id_A) = Id_{F(A)}, \quad F(f \circ g) = F(g) \circ F(f))$

Given M an R-module,

$F = \mathrm{Hom}_R(M, -)$ is a covariant functor $R\text{-Mod} \to Ab$

$G = \mathrm{Hom}_R(-, M)$ is a contravariant functor (R-Mod R-comm)

$F(N) = \mathrm{Hom}_R(M,N)$, $\quad G(N) = \mathrm{Hom}_R(N,M)$

If $f : N \to N'$, $\quad F(f) = f_*$

$\mathrm{Hom}_R(N,N') \longrightarrow \mathrm{Hom}(F(N), F(N'))$

$f_*(g : M \to N) = (f \circ g) : M \to N'$

$G(f) = f^*$

$f^*(g : N' \to M) = g \circ f : N \to M$

f_* and f^* are homomorphisms of abelian groups

<u>Thm</u> If $0 \to N \xrightarrow{f} M \xrightarrow{g} N'$ is exact, then

$0 \to \mathrm{Hom}_R(M', N) \xrightarrow{f_*} \mathrm{Hom}_R(M', M) \xrightarrow{g_*} \mathrm{Hom}_R(M', N')$ is

If $N \xrightarrow{f} M \xrightarrow{g} N' \to 0$ is exact, then

$0 \to \mathrm{Hom}_R(N', M') \xrightarrow{g^*} \mathrm{Hom}_R(M, M') \xrightarrow{f^*} \mathrm{Hom}_R(N, M')$ is exact.

<u>Def</u> Hom is a left exact functor.

<u>Pf</u> $\ker(f_*) = 0$

$\mathrm{Im}(f_*) = \ker(g_*)$

$$h \in \ker(f_*) \longleftrightarrow f_*(h) = 0 : M' \longrightarrow M$$

$$\begin{aligned} &\quad\ \, \| \\ &f \circ h = 0 : M' \longrightarrow M \\ &\qquad \Updownarrow \end{aligned}$$

$$f(h(m')) = 0 \quad \text{for all} \quad m' \in M$$

$$f \text{ injective} \Rightarrow h(m') = 0 \text{ for all } m' \in M$$

$$\Rightarrow h = 0$$

$\mathrm{Im}(f_*) \subseteq \ker(g_*)$:

$$g_*(f_*(h)) = g \circ (f \circ h) = (\underset{\substack{\| \\ 0}}{g \circ f}) \circ h$$

If $h : M' \longrightarrow M$ is in $\ker(g_*)$, then

$h = f \circ \bar{h}$, $\bar{h} : M' \longrightarrow N$. $\longleftrightarrow$ $\ker(g_*) \subseteq \mathrm{Im}(f_*)$

$$\begin{tikzcd} M' \arrow[r, "h"] \arrow[rr, bend right] & M \arrow[r, "g"] & N' \end{tikzcd}$$

$$g_*(h) = 0. \Rightarrow \forall m', \quad h(m') \in \underset{\substack{\| \\ \mathrm{Im}(f)}}{\ker(g)} \text{ by exactness.}$$

On $\mathrm{Im}(f)$, there is an inverse for f, so we can compose h w/ this inverse, getting $\bar{h}$.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] & M \arrow[r, "g"] & N' \\ & & M' \arrow[ul, dashed, "\bar{h}"] \arrow[u, "h"] \arrow[ur, "g \circ h = 0"'] & \end{tikzcd}$$

$\boxtimes$