Lecture 16 – Isomorphism Theorems

Note Title 3/20/2008

<u>Remark</u>: If $f: M \longrightarrow N$ an R-mod hom, then $\ker(f)$ is a submodule.

<u>Thm</u> If $f: M \to N$ is an R-mod hom, then f induces an isomorphism $M/_{\ker(f)} \longrightarrow \mathrm{Im}(f)$.

<u>Pf</u> f is a hom of abelian groups $\Rightarrow$ get a map $\tilde{f}: M/_{\ker(f)} \xrightarrow{\cong} \mathrm{Im}(f)$

$$\tilde{f}([m]) = f(m).$$

$$\tilde{f}(r \cdot [m]) = \tilde{f}([r \cdot m]) = f(r \cdot m) = r \cdot f(m) = r \cdot \tilde{f}([m]). \quad \boxtimes$$

<u>Thm</u>: $N \subseteq N' \subseteq M$ then

$$M/_{N'} \cong (M/N)/_{(N'/N)}$$

If N, N' are submodules of M, then can define $N + N'$ to be $\{n + n' \mid n \in N, n' \in N'\}$. This is a submodule of M.

$$(n_1 + n_1') + (n_2 + n_2') = \underbrace{(n_1 + n_2)}_{\in N} + \underbrace{(n_1' + n_2')}_{\in N'} \in N + N'$$

$$r \cdot (n + n') = \underset{\in N}{(r \cdot n)} + \underset{\in N'}{(r \cdot n')}$$

$$N, N' \subseteq N + N'$$

$N' \mapsto 0 + N'$, $N \mapsto N + 0$

$N + N'$ is the smallest submodule containing $N \cup N'$!

<u>Thm</u>: $N + N'/_{N'} \cong N/_{N \cap N'}$

<u>Pf</u>:

$$\begin{aligned} N &\longrightarrow N + N'/_{N'} \\ n &\longmapsto [n] \end{aligned} \quad \boxtimes$$

<u>Thm</u>: There is a 1-1 correspondence between submodules of M containing N and submodules of M/N.

<u>Def</u> – Let M be an R-module, then the submodule gen by $\{m_1, \dots\}$ is the intersection of all submodules containing it.

- If there is a finite set $\{m_1, \dots, m_n\}$ such that $M = \text{Span}(m_1, \dots, m_n)$, then M is finitely generated.
- If $M = \text{Span}(m_1)$, M is cyclic.
- If M is f.g., then $\text{rank}(M)$ is the least number of generators.

Given two R-modules M & N, can form their direct sum:

$$M \oplus N = \{(m,n) \mid m \in M, n \in N\} \quad \text{w/ coordinatewise ops.}$$

Given a family of R-modules M_i, $i \in I$ can form

1) $\prod_{i \in I} M_i = \{f : I \to \bigcup M_i \mid f(i) \in M_i\}$

w/ coord. ops: $(f+g)(i) = f(i) + g(i)$

2) $\bigoplus_{i \in I} M_i = \{f : I \to \bigcup M_i \mid f(i) \in M_i,\ f(i) = 0 \text{ for almost all } i\}$

$\prod_{i \in I} M_i$ is big. but $\prod_{i \in I} M_i \supseteq \bigoplus_{i \in I} M_i$ & $M_i \subseteq \bigoplus_{i \in I} M_i$

Then $\bigoplus M_i = \langle M_1, \dots, M_i, \dots \rangle$

R is an R-module

$$\underset{1 \mapsto a}{\text{Hom}_{Ab}(\mathbb{Z}, A) = A} \longleftrightarrow \underset{1 \longmapsto m}{\text{Hom}_{R\text{-mod}}(R, M) = M}$$

$$f(r) = f(r \cdot 1) = r \cdot f(1)$$

If $\{m_1, \dots, m_n\}$ is a set of generators of M, then have a surjective map

$$\underbrace{R \oplus \dots \oplus R}_{n} \longrightarrow M$$

$$(0, \dots, 1, \dots, 0) \longmapsto m_i$$

$$(a_1, \dots, a_n) = a_1(1, \dots, 0) + a_2(0, 1, \dots 0) + \dots$$

$$\downarrow \qquad\qquad \downarrow$$

$$a_1 m_1 + a_2 m_2 + \dots + a_n m_n$$

$\Rightarrow$ surjective

$\Rightarrow \quad M = (R \oplus \dots \oplus R)/\ker$

If M is cyclic, then M has 1 gen $\Rightarrow$ M is $R/\ker(\,)$

$\ker(\,)$ is a left ideal of R.

Def: If $X \subseteq M$ is a subset, then

$$\mathrm{Ann}(X) = \{r \in R \mid r \cdot x = 0,\ x \in X\}$$

$\mathrm{Ann}(x)$ is a left ideal: $r \in \mathrm{Ann}(x)$, $s \in R$, then

$$(s \cdot r) \cdot x = s \cdot (r \cdot x) = s \cdot 0 = 0$$

If X is a submodule, then $\mathrm{Ann}(X)$ is an ideal in R.

$$(r \cdot s) \cdot x = r \cdot (\underset{X}{\underset{\cap}{s \cdot x}}) = 0 \Rightarrow r \cdot s \in \mathrm{Ann}(X).$$

Ex F a field, V an F-vector space. Given $T: V \to V$, get an $F[x]$-module structure on V by

$$(a_0 + \dots + a_n x^n) \cdot \bar{v} = a_0 \bar{v} + a_1 T(\bar{v}) + \dots + a_n T^n(\bar{v}).$$

$$\leftrightarrow \quad F[x] \longrightarrow \mathrm{End}(V)$$
$$x \longmapsto T$$

$F = \mathbb{R}$ $\quad V = \mathbb{R}^2$ $\quad T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\mathbb{R}[x] \longrightarrow \mathrm{End}(V) = M_2(\mathbb{R})$$
$$x \longmapsto T$$

$T^2 + 1 = 0 \Rightarrow x^2 + 1$ is in the kernel of $\mathbb{R}[x] \longrightarrow \mathrm{End}(V)$.

In fact $\langle x^2+1 \rangle = \ker(\ )$.

char poly of T.

Def: An R module is free if it is isomorphic to $\bigoplus_{i \in I} R$ some I.

$\leftrightarrow$ can find a basis for our module

(a set that spans & is lin. ind).

Ex: • $\mathbb{Z}/n$ as a $\mathbb{Z}$-mod:

has no lin. ind sets.

$m \in \mathbb{Z}/n$ $\quad \{m\}$ is not lin. ind: $\quad n \cdot m = 0$

• $\mathbb{Q}$ is not finitely generated

$\{\frac{p}{q}, \frac{r}{s}\}$ is lin. dep

$$(rq) \cdot \left(\frac{p}{q}\right) - (ps) \cdot \left(\frac{r}{s}\right) = 0 \quad \Rightarrow \mathbb{Q} \text{ is not free}$$

- $\mathbb{Q}/\mathbb{Z} = \left\{ \frac{p}{q} \mid |p| < |q| \right\}$

  is not f.g & has no lin ind subsets.

Say $x \in M$ is torsion if $Ann(x) \neq \{0\}$.

Say M is torsion free if $\forall x \in M$, $Ann(x) = \{0\}$.

Define $M_\tau = \{ m \in M \mid Ann(m) \neq \{0\} \}$ = torsion submodule

If R is an integral Domain, then M_τ is a submodule.

$m \in M_\tau$, $\quad r \cdot m \overset{?}{\in} M_\tau$

$\Downarrow$

$\exists s.$ s.t

$s \cdot m = 0 \Rightarrow s \cdot (r \cdot m) = (s \cdot r) \cdot m = (r \cdot s) \cdot m = r \cdot (s \cdot m) = 0$

$m, n \in M_\tau$

$\Downarrow$

s_m, s_n s.t.

$s_m \cdot m = s_n \cdot n = 0$

$\Rightarrow (s_m \cdot s_n)(m+n) = 0$