Lecture 15 – Modules

Note Title 3/18/2008

Def Let R be a unital ring, a left R-module is an abelian group M together with an action: $\cdot : R \times M \longrightarrow M$ (scalar mult) satisfying:

1) $a\cdot(b\cdot m) = (a\cdot b)\cdot m$
2) $1\cdot m = m$
3) $(a+b)\cdot m = a\cdot m + b\cdot m$
4) $a(m+n) = a\cdot m + a\cdot n$

Remark: If R is a field (division Ring) left R-mod $=$ R-vector space.

Remark: A right R-module is defined similarly: $\cdot\ M \times R \longrightarrow M$

1') $(m\cdot a)\cdot b = m\cdot(ab)$.

A right R-mod $\longleftrightarrow$ left module over R^{op}: $R, +, *$

$$a * b = b\cdot a$$

M is a left R^{op} Mod:

$$(a*b)\cdot m = a\cdot(b\cdot m)$$
$$\overset{\shortparallel}{(b\cdot a)}\cdot m$$

Ex: $\mathbb{Z}$-module $\longleftrightarrow$ abelian group

$M \longrightarrow M$ (forget $\mathbb{Z}$-action)

$\mathbb{Z}\times A \to A \longleftarrow A$

$$n\cdot a = \begin{cases} \underbrace{a+a+\dots+a}_{n} & n>0 \\ 0 & n=0 \\ \underbrace{-a-a-a\dots-a}_{-n} & n<0 \end{cases}$$

$n\cdot a = (1+(n-1))a$

$= a + (n-1)a$

$0\cdot a = 0$

$-1\cdot a = -a$

Def: A homomorphism of R-modules $f: M \to N$ is a hom of the underlying abelian groups s.t.

$$f(r \cdot m) = r \cdot f(m) \quad \text{for all } r \in R,\ m \in M.$$

Ex R a field, R-mod homs $\leftrightarrow$ linear transformations.

Prop - If M, N are R-mod, then $\mathrm{Hom}_R(M,N)$ is an abelian group under $(f+g)(m) = f(m) + g(m)$.

- If R is commutative, $\mathrm{Hom}_R(M,N)$ is an R-mod
$(af)(m) = a \cdot (f(m))$.

Pf: - $(f+g)(am+bn) = f(am+bn) + g(am+bn)$

$$= (af(m) + bf(n)) + (ag(m) + bg(n))$$
$$= af(m) + ag(m) + bf(n) + bg(n)$$
$$= a(f(m) + g(m)) + b(f(n) + g(n))$$
$$= a \cdot (f+g)(m) + b \cdot (f+g)(n)$$

- $(af)(bm) \overset{?}{=} b((af)(m)) = (b \cdot a) \cdot f(m)$

$\quad \|$

$a \cdot f(bm) = (a \cdot b) f(m)$

R com $\Rightarrow a \cdot b = b \cdot a \Rightarrow af$ is R-linear. $\boxtimes$

Remark: Let $Z(R)$ be the center of the ring $= \{a \mid a \cdot b = b \cdot a \ \forall\, b \in R\}$

$\Rightarrow \mathrm{Hom}_R(M,N)$ is a $Z(R)$-module, for all R, M, N.

Another def of left R-mod:

For any abelian group A, $\mathrm{End}(A) = \{f: A \to A \mid f \text{ hom}\}$ is a ring:

$$(f+g)(a) = f(a) + g(a)$$
$$(f \cdot g)(a) = f(g(a)).$$

- $\cdot$ is associative
- $\mathrm{Id}: A \to A$ is the identity in $\mathrm{End}(A)$
- $(f \cdot (g+h))(a) = f((g+h)(a)) = f(g(a)+h(a)) = f(g(a)) + f(h(a))$
$= (f \cdot g)(a) + (f \cdot h)(a)$

$\Rightarrow f \cdot (g+h) = f \cdot g + f \cdot h$

Moral Example: $V = \mathbb{R}^n$, $\mathrm{End}_{\mathbb{R}}(V) \cong M_{n\times n}(\mathbb{R})$

An R-module is an abelian group M together with a ring hom
$R \to \mathrm{End}(M) \leftarrow$ "is" the scalar mult.

$R \times M \longrightarrow M$

for all $a \in R$, $\quad a\cdot(-) : M \longrightarrow M$

$$a\cdot(m+n) = a\cdot m + a\cdot n \Rightarrow a\cdot(-) \in \mathrm{End}(M)$$

$\Rightarrow$ function $\quad R \longrightarrow \mathrm{End}(M)$

$$a \longmapsto a\cdot(-) = l_a$$

$$a+b \longmapsto (a+b)\cdot(-)$$

$$(a+b)\cdot(m) = a\cdot m + b\cdot m$$

$$\Rightarrow \underset{l_{a+b}}{\underset{\parallel}{(a+b)\cdot(-)}} = \underset{l_a + l_b}{\underset{\parallel}{a\cdot(-) + b(-)}}$$

$$a\cdot b \longmapsto (a\cdot b)\cdot(-)$$

$$\underset{l_{ab}(m)}{\underset{\parallel}{(a\cdot b)(m)}} = a\cdot(b\cdot m) = a\cdot l_b(m) = l_a(l_b(m)) \Rightarrow l_{ab} = l_a \circ l_b .$$

Given $f: R \to \mathrm{End}(M)$

define $R \times M \longrightarrow M$ by

$$a\cdot m = (f(a))(m)$$

$$\mathrm{Bilin}(M,N;P) = \{f : M\times N \to P \mid f \text{ bilinear}\}$$

$$\mathrm{Bilin}(M,N;P) = \mathrm{Hom}(M, \mathrm{Hom}(N,P))$$

$$\mathrm{Bilin}(M,N;P) = \mathrm{Hom}(M\otimes N, P)$$

Def A submodule of M is a subgroup N that's closed under scalar mult: N a submod iff $r\cdot n + s\cdot m \in N$, whenever $n, m \in N$.

Ex: R is an R-module. A left ideal = submodule for left R-mod

A right ideal = " " Right mod structure

An ideal = both.

Def If $N \subseteq M$ is a submodule, then M/N inherits the structure of an R-mod

$$r(m+N) = (rm)+N.$$

$$r(m+n) = r\cdot m + r\cdot n \in r\cdot m + N$$

Def If $f: M \to N$ an R-mod hom, then $\ker(f) = \{m \mid f(m) = 0\}$.

Thm $M \longrightarrow M/N$ is an R-mod map, and if $f: M \to P$ is such that $\ker(f) \supseteq N$, then $\exists$ a unique map : $\tilde{f}: M/N \to P$: s.t.

$$\begin{tikzcd} M \arrow[d] \arrow[dr, "f"] & \\ M/N \arrow[r, dashed, "\tilde{f}"'] & P \end{tikzcd}$$