Lecture 13 - Polynomial Rings & Examples

Note Title 2/28/2008

Finished with

$R \times S/\sim \qquad (a,s)\sim(b,t) \quad \text{iff} \quad at=bs$

$$\frac{a}{s} = \frac{b}{t}$$

+) $\quad \frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$

·) $\quad \frac{a}{s}\cdot\frac{b}{t} = \frac{ab}{st}$

$\frac{a'}{s'} \sim \frac{a}{s}, \quad \frac{b'}{t'} \sim \frac{b}{t} \quad \longleftrightarrow \quad a's - as' = 0, \quad b't - bt' = 0$

$$\left(\frac{a}{s} + \frac{b}{t}\right) \overset{?}{\sim} \left(\frac{a'}{s'} + \frac{b'}{t'}\right)$$

$$\frac{at+bs}{st} \overset{?}{\sim} \frac{a't'+b's'}{s't'} \longleftrightarrow ats't' + bss't' \overset{?}{=} a't'st + b's'st$$

$$0 \overset{?}{=} (a'stt' - as'tt') + (b'tss' - bt'ss')$$

$$= (a's - as')tt' + (b't - bt')ss'$$

$$= 0\cdot tt' + 0\cdot ss'.$$

$\Rightarrow ((R\times S)/\sim, +, \cdot)$ is a ring: $R_S = R[S^{-1}]$

$$0 = \frac{0}{s}, \quad -\frac{a}{s} = \frac{-a}{s}, \quad 1 = \frac{s}{s}$$

In R_S, everything in S is a unit: $\quad \frac{s^2}{s}\cdot\frac{s}{s^2} = \frac{s^3}{s^3} = 1$

$R \xrightarrow{\lambda} R_S \qquad \lambda(r) = \frac{rs}{s}$

$\lambda(1) = \frac{s}{s} = 1_{R_S}$

$\lambda(a+b) = \frac{(a+b)s}{s} = \frac{as+bs}{s} = \frac{as\cdot s + bs\cdot s}{s\cdot s} = \frac{as}{s} + \frac{bs}{s} = \lambda(a)+\lambda(b)$

$$\lambda(ab) = \frac{abs}{s} = \frac{abs^2}{s^2} = \frac{(as)(bs)}{s^2} = \frac{as}{s}\cdot\frac{bs}{s} = \lambda(a)\cdot\lambda(b).$$

λ is injective: $\quad r \mid \lambda(r) = \frac{rs}{s} = 0 = \frac{0}{t}$

$$\Rightarrow \frac{rs}{s} = \frac{0}{t} \longleftrightarrow r\underbrace{st}_{\in S} = 0\cdot s = 0$$

$$\Rightarrow r = 0.$$

Prop 55: $R \xrightarrow{\lambda} R_S$ satisfies the universal property: $R \xrightarrow{f} R'$ s.t. $f(s)$ is a unit $\forall s \in S$, then there is a unique homomorphism $R_S \xrightarrow{\tilde{f}} R'$ s.t.

$$\begin{tikzcd} R \arrow[d, "\lambda"'] \arrow[dr, "f"] & \\ R_S \arrow[r, "\tilde{f}"'] & R' \end{tikzcd}$$

Pf: $\quad \tilde{f}\left(\frac{a}{s}\right) = \frac{f(a)}{f(s)}$

if $\frac{a'}{s'} \sim \frac{a}{s}$, then $a's - as' = 0 \Rightarrow f(a's - as') = 0$

$$\Rightarrow f(a')f(s) - f(a)f(s') = 0$$

$$\Rightarrow f(a')f(s) = f(a)f(s')$$

$$\Rightarrow \frac{f(a')}{f(s')} = \frac{f(a)}{f(s)} \Rightarrow \tilde{f}\left(\frac{a'}{s'}\right) = \tilde{f}\left(\frac{a}{s}\right).$$

$$(\tilde{f}\circ\lambda)(r) = \tilde{f}\left(\frac{rs}{s}\right) = \frac{f(rs)}{f(s)} = \frac{f(r)\cancel{f(s)}}{\cancel{f(s)}} = f(r). \qquad \boxtimes$$

Ex: • $R = \mathbb{Z}$, $\quad S = \mathbb{Z} - \{0\}$. $\Rightarrow R_S = \mathbb{Q}$.

• R is an integral domain, $S = R - \{0\} \Rightarrow R_S$ is $Q(R) = F(R)$, the field of fractions.

• $R = \mathbb{Z}$, $\quad S = \{n, n^2, \dots\}$, $\quad R_S = \mathbb{Z}[1/n] = \{\frac{a}{n^k} \mid a \in \mathbb{Z},\ k \in \mathbb{N}\}$

• $R = \mathbb{Z}$, $\quad S = R - (p)$, $\quad R_S = \mathbb{Z}_{(p)} = \{\frac{a}{b} \mid b$ is relatively prime to $p\}$

R_S has one one nontrivial, proper, prime ideal (a maximal ideals): (p). $\quad R_S$ is a local ring

R is a unital ring.

Def The polynomials in an indeterminant x over R are $R[x] =$

$$\{a_0 + a_1 x + \dots + a_n x^n \mid a_i \in R,\ n \in \mathbb{N}\}$$

Mult & addition is as usual: FOIL out products.

Remark: $R[x] = \{f : \mathbb{N} \to R \mid \exists N \text{ s.t. } f(n) = 0 \ \forall n > N\}$

$$\left.\begin{aligned} (f \cdot g)(n) &= f(n) \cdot g(n) \\ (f+g)(n) &= f(n) + g(n) \end{aligned}\right\} \text{pointwise operations.}$$

If $R \xrightarrow{\phi} S$ is a ring hom, then get a ring hom $R[x] \xrightarrow{\phi} S[x]$

$$\phi(a_0 + \dots + a_n x^n) = \phi(a_0) + \dots + \phi(a_n) x^n$$

If all rings are commutative, then $R[x]$ has a universal property:

$$\mathrm{Hom}_{\text{Rings}}(R[x], S) = S$$

(extends $R \to S$)

$$R \hookrightarrow R[x]$$
$$a \longmapsto a$$

$R \xrightarrow{\phi} S$ $\quad (S = R,\ \phi = \mathrm{Id})$

$$\mathrm{Hom}(R[x], R) = R$$
$$\big(\underbrace{p(x) \mapsto p(a)}_{\phi_a}\big) \longleftrightarrow a$$

Given $p \in R[x]$, $\quad \phi_a(p) = p(a)$.

I. R unital, $M_n(R) = \{n \times n \text{ matrices} / R \text{ w/ matrix } + \text{ \& matrix } \cdot\}$

$$\cup\!\!\mid$$

$$(M_n(R))^* = GL_n(R)$$

II. If G is a group, $R(G)$ = group ring

$$= \Big\{\sum_{g \in G} r_g \cdot g \;\Big|\; r_g \in R,\ r_g = 0 \text{ all but finitely many } g\Big\}$$

$$(r_g \cdot g) \cdot (r_h \cdot h) = (\underbrace{r_g r_h}_{\text{in ring}}) \cdot (\underbrace{gh}_{\text{in } G}).$$

$R = \mathbb{Z}, \quad G = \mathbb{Z}/2 = \{e, g\}$

$R(G) = \{ n \cdot e + m \cdot g \}$

$$(n \cdot e + m \cdot g)(r \cdot e + s \cdot g) = (n \cdot e)(r \cdot e) + (m \cdot g) \cdot (r \cdot e) + (n \cdot e) \cdot (s \cdot g) + (m \cdot g) \cdot (s \cdot g).$$

$$= (nr) \cdot e + (mr) \cdot g + (ns) \cdot g + (ms) \cdot e$$

$$= (nr + ms) e + (mr + ns) g$$

$$\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}(G), S) = \mathrm{Hom}_{\mathrm{Grp}}(G, S^{\times}).$$