Def. The ideal/subring generated by a set \( X \) is the intersection of all ideals/subrings containing \( X \).

Prop 51: 1) Subring generated by \( X \) is the collection of all sums/differences of products of elements in \( X \).

2) \( \langle X \rangle = R \times X = \sum_{i=1}^{n} r_i x_i s_i \mid r_i, s_i \in R, x_i \in X \)

3) If \( R \) is commutative,

\[ \langle X \rangle = RX = \sum_{i=1}^{n} r_i x_i \mid r_i \in R, x_i \in X \]

Def. An ideal \( I \) is maximal if \( I \subseteq J \subseteq R \), \( J \) an ideal \( \Rightarrow J = I \) or \( J = R \).

An ideal \( I \) is prime if \( ab \in I \), then \( a \in I \) or \( b \in I \).

Thm 52: (If \( R \) is commutative) every ideal is contained in a maximal ideal.

Recall: \( a \leq b \): 1) \( a \leq a \)

2) \( a \leq b, b \leq a \Rightarrow a = b \)

3) \( a \leq b, b \leq c \Rightarrow a \leq c \)

On \( \sum I \mid I \) is a proper ideal, \( \leq \) is a partial order.

Zorn's Lemma: In a partially ordered set, if every chain has a least upper bound, then there are maximal elements.

Chain: \( \ldots \leq a \leq a_{n+1} \leq \ldots \)

Least ub: \( C \) a chain, \( u \) is an lub if \( a \leq u \) if \( v \) has the same property, then \( u \leq v \).

Pf: \( \ldots \subseteq I_1 \subseteq I_2 \subseteq \ldots \) is a chain in \( \sum I \mid I \) proper, \( \leq \)

The least upper bound: \( U_{\sum I} \),

\[ x \in U_{\sum I} \Rightarrow x \in I_n \quad y \in U_{\sum I} \Rightarrow y \in I_n \]

\[ x, y \in I_n \Rightarrow rx, xr, xy, x-y \in I_n \quad (I_n \text{ an ideal}) \]

\[ \Rightarrow rx, xr, xy, etc \in U_{\sum I} \]

1 \( \notin I_n \) for any \( n \), since \( I_n \) is proper, \( 1 \notin U_{\sum I} \Rightarrow U_{\sum I} \) proper.
Zorn's Lemma implies maximal elements.

Ex: Maximal ideals in $\mathbb{Z}$ are $\langle p \rangle = p\mathbb{Z} = \mathbb{Z}/p$

= prime ideals.

Thm 53: If $M$ is maximal, then $R/M$ is a field
- If $R/M$ is a field, then $M$ is maximal.
- If $R$ is prime, " " " " , then $R/M$ is an integral domain.
- If $R/\mathfrak{a}$ is an integral domain, $\mathfrak{a}$ prime.

Pf: $M$ maximal: Choose $a \neq 0 \in R/M$. Look at the ideal gen by $a$.
- Lift to an ideal in $R$ via $R \xrightarrow{\pi_M} R/M$
  $$\pi_M^{-1}(\langle a \rangle) = Ra + M$$ is an ideal in $R$ that contains $M \Rightarrow Ra + M = M$ or $Ra + M = R$ ($M$ maximal)
  $a \notin M \Rightarrow Ra + M$ $\notin M \Rightarrow Ra + M = R$
  $\Rightarrow$ have $b \in R$ s.t. $m \in M$ s.t.
  $$ba + m = 1.$$ 
  $$\Rightarrow \pi_M(ba + m) = 1$$
  $$\pi_M(b) \pi_M(a) + \pi_M(m) = \pi_M(b), \pi_M(a)$$
  $$\Rightarrow \pi_M(b)$$ is the desired inverse.

If $R/M$ is a field, then $R/M$ has exactly 2 ideals: $\mathbb{Z}_0$ $\neq R/M$.
- $(a \neq 0, a \in I, \text{then } 1 = a^{-1}a \in I \Rightarrow I = R/M)$ (holds in general)
  $\Rightarrow$ the only ideals between $M$ and $R$ are $M$ and $R$. $\Rightarrow M$ max.

If $A$ prime: $\bar{a}, \bar{b} \in R/\mathfrak{a}, \text{then } \bar{a}\bar{b} = \bar{0}$, $\text{then } a \cdot b \in \mathfrak{a}$.
  $\Rightarrow$ either $a \in \mathfrak{a}$ or $b \in \mathfrak{a}$ $\Rightarrow \bar{a}$ or $\bar{b} = \bar{0}$. $\Rightarrow R/\mathfrak{a}$ has no zero divisors.

Converse follows by reversing arrows. \(\square\)

Cor 54: Maximal ideals are prime (R is comm).

Def A multiplicative subset $S$ is one for which $a, b \in S \Rightarrow ab \in S$ ($S$ contains no zero divisors)
Ex: If $R$ is an integral domain, $R - \{0\}$ is a mult-subset.
- If $f \in R$ is not a zero divisor, then $2f, f^2, f^3, \ldots \in R$ is a mult-subset.
- If $\pi$ is prime, then $R - \pi$ is a mult-subset.
- If $S_1 \checkmark S_2$ are mult, then so is $S_1 \cap S_2$.

**Def** The localization of $R$ away from $S$ is the ring $R_S$ together with a map $R \rightarrow R_S$ that satisfies the following universal property: If $R \xrightarrow{f} R'$ is a homomorphism s.t.

$$f(s) \in (R')^X \quad \forall s \in S,$$

then there is a unique map $\tilde{f} : R_S \rightarrow R'$.

$$\begin{array}{ccc}
R & \rightarrow & R'\\
\downarrow & \searrow & \searrow \tilde{f} \\
R_S & \rightarrow & R'
\end{array}$$

Construction: Look at fractions $\frac{a}{s}$, $a \in R$, $s \in S$.

1) $R \times S$ put on this an equivalence relation: $(a, s) \sim (b, t)$ iff $at = bs$.

$(a, s) \sim (b, t), \quad (b, t) \sim (c, u)$

$at = bs \quad bu = ct$

$atu = bsu = sbu = sct \Rightarrow atu = sct$

$\Rightarrow t(au - sc) = 0.$

$\Rightarrow au - sc = 0. \quad \Leftrightarrow (a, s) \sim (c, u).$

2) $R \times S/\sim$:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$

$$\frac{a}{s} \cdot \frac{b}{t} = \frac{a \cdot b}{s \cdot t}.$$