Lecture 12 - Ideals & Localizations

Note Title 2/26/2008

Def The ideal / subring generated by a set X is the intersection of all ideals / subrings containing X. $\langle x \rangle$

Prop 51: 1) Subring gen by X is the collection of all sums / differences of products of elements in X.

R has 1. 2) $\langle x \rangle = RXR = \{ \sum_{i=1}^{n} r_i x_i s_i \mid r_i, s_i \in R, x_i \in X \}$

3) If R is commutative,

$$\langle x \rangle = RX = \{ \sum_i r_i x_i \mid r_i \in R \quad x_i \in X \}$$

Def • An ideal I is maximal if $I \subseteq J \subseteq R$, J an ideal $\Rightarrow J = I$ or $J = R$.

• An ideal P is prime if $ab \in P$, then $a \in P$, or $b \in P$.

Thm 52: (If R is commutative) every ideal is contained in a maximal ideal.

Recall: $a \leq b$: 1) $a \leq a$

2) $a \leq b, \ b \leq a \Rightarrow a = b$

3) $a \leq b, \ b \leq c, \Rightarrow a \leq c$

On $\{ I \mid I$ is a proper ideal$\}$, $\subseteq$ is a partial order.

Zorn's Lemma: In a partially ordered set, if every chain has a least upper bound, then there are maximal elements.

chain: $\ldots \leq a_i \leq a_{i+1} \leq \ldots$

least ub: $\mathcal{C}$ a chain, u is an lub if $a_i \leq u$ & if v has the same property, then $u \leq v$.

Pf: $\ldots \subseteq I_n \subseteq I_{n+1} \subseteq \ldots$ is a chain in $\{ I \mid I$ proper$\}$, $\subseteq$

The least upper bound: $\bigcup_{n \in \mathbb{Z}} I_n$.

$x \in \bigcup I_n \Rightarrow x \in I_m$
$y \in \bigcup I_n \Rightarrow y \in I_k$ $\Rightarrow$ $x, y \in I_n \Rightarrow rx, xr, x+y, x-y \in I_n$ (I_n an ideal)

$\Rightarrow rx, xr, x+y,$ etc $\in \bigcup I_n$.

$1 \notin I_n$ for any n, since I_n is proper. $\Rightarrow 1 \notin \bigcup I_n \Rightarrow \bigcup I_n$ proper.

Zorn's Lemma $\Rightarrow$ have maximal elements. $\boxtimes$

Ex: Maximal ideals in $\mathbb{Z}$ are $\langle p \rangle = p\mathbb{Z} = \mathbb{Z}p$

$=$ prime ideals.

Thm 53 • If $\mathfrak{m}$ is maximal, and R comm, then $R/\mathfrak{m}$ is a field

• If $R/\mathfrak{m}$ is a field, then $\mathfrak{m}$ is maximal.

• If $\mathfrak{p}$ is prime, " " ", then $R/\mathfrak{p}$ is an integral domain.

• If $R/\mathfrak{p}$ is an integral domain, $\mathfrak{p}$ prime.

Pf: $\mathfrak{m}$ maximal: Choose $a \neq 0 \in R/\mathfrak{m}$. Look at the ideal gen by a:

$\langle a \rangle$ & lift to an ideal in R via $R \xrightarrow{\pi_{\mathfrak{m}}} R/\mathfrak{m}$

$\pi_{\mathfrak{m}}^{-1}(\langle a \rangle) = Ra + \mathfrak{m}$ is an ideal in R that contains

$\mathfrak{m} \Rightarrow Ra + \mathfrak{m} = \mathfrak{m}$ or $Ra + \mathfrak{m} = R$ ($\mathfrak{m}$ maximal)

$a \notin \mathfrak{m} \Rightarrow Ra + \mathfrak{m} \notin \mathfrak{m} \Rightarrow Ra + \mathfrak{m} = R$

$\Rightarrow$ have $b \in R$ & an $m \in \mathfrak{m}$ s.t.

$$ba + m = 1.$$

$$\Rightarrow \pi_{\mathfrak{m}}(ba + m) = 1$$

$$\pi_{\mathfrak{m}}(b)\,\pi_{\mathfrak{m}}(a) + \pi_{\mathfrak{m}}(m) = \pi_{\mathfrak{m}}(b) \cdot \pi_{\mathfrak{m}}(a)$$

$\Rightarrow \pi_{\mathfrak{m}}(b)$ is the desired inverse.

If $R/\mathfrak{m}$ is a field, then $R/\mathfrak{m}$ has exactly 2 ideals: $\{0\}$ & $R/\mathfrak{m}$.

($a \neq 0$, $a \in I$, then $1 = a^{-1}a \in I \Rightarrow I = R/\mathfrak{m}$) (holds in general)

$\Rightarrow$ the only ideals between $\mathfrak{m}$ and R are $\mathfrak{m}$ and R. $\Rightarrow \mathfrak{m}$ max.

If $\mathfrak{p}$ prime: $\bar{a}, \bar{b} \in R/\mathfrak{p}$, then $\bar{a}\cdot\bar{b} = \bar{0}$, then $a \cdot b \in \mathfrak{p}$.

$\Rightarrow$ either $a \in \mathfrak{p}$ or $b \in \mathfrak{p} \Rightarrow \bar{a}$ or $\bar{b} = 0$. $\Rightarrow R/\mathfrak{p}$ has no zero divisors.

Converse follows by reversing arrows. $\boxtimes$

Cor 54: Maximal ideals are prime (R is comm).

Def A multiplicative subset S is one for which $a, b \in S \Rightarrow ab \in S$

(S contains no zero divisors)

Ex: - If R is an integral domain, $R - \{0\}$ is a mult subset.

- If $f \in R$ is not a zero divisor, then $\{f, f^2, f^3, \dots\}$ is a mult sub.
- If $\mathfrak{p}$ is prime, then $R - \mathfrak{p}$ is a mult subset.
- If S_1 & S_2 are mult, then so is $S_1 \cap S_2$.

Def The localization of R away from S is the ring R_S together with a map $R \to R_S$ that satisfies the following universal property: If $R \xrightarrow{f} R'$ is a homomorphism s.t. $f(s) \in (R')^\times$ $\forall s \in S$, then there is a unique map $\tilde{f}: R_S \to R'$:

$$\begin{tikzcd} R \arrow[d] \arrow[dr, "f"] & \\ R_S \arrow[r, "\tilde{f}"'] & R' \end{tikzcd}.$$

Construction: Look at fractions $\frac{a}{s}$, $a \in R$, $s \in S$.

1) $R \times S$ & put on this an equiv relation: $(a,s) \sim (b,t)$ iff

$$at = bs.$$

$$(a,s) \sim (b,t), \quad (b,t) \sim (c,u)$$

$$at = bs \qquad bu = ct$$

$$atu = bsu = sbu = sct \Rightarrow atu = sct$$

$$\Rightarrow t(au - sc) = 0.$$

$$\Rightarrow au - sc = 0. \longleftrightarrow (a,s) \sim (c,u).$$

2) $R \times S/\sim$: $+, \cdot$

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$

$$\frac{a}{s} \cdot \frac{b}{t} = \frac{a \cdot b}{s \cdot t}.$$