Lecture 10 – Symmetric Groups II & Rings

Note Title 2/19/2008

Def The sign of an element of S_n is the number of transpositions required to get that element, mod 2.

$$\sigma(f) = \begin{cases} 1 & f \text{ is a prod of an even number of trans} \\ -1 & f \text{ is a prod of an odd number of trans.} \end{cases}$$

Prop 38 This is a well-defined homomorphism $S_n \longrightarrow \mathbb{Z}/2$.

Pf: $\alpha \in S_n$, $\quad f(\alpha) = \sum_i (o(\alpha_i) - 1)$, where $\alpha = \alpha_1 \cdots \alpha_k$ is the decomp into disjoint cycles.

Need to show $f(\alpha(ab)) \equiv f(\alpha) + 1 \mod 2$

$f(\mathrm{Id}) = 0$ & if $\alpha = (a_1 b_1) \cdots (a_n b_n)$, then

$$0 = f(\mathrm{Id}) = f(\alpha \cdot (a_n b_n) \cdots (a_1 b_1)) = f(\alpha) + n \quad (2)$$

$$\Rightarrow f(\alpha) = n \quad (2)$$

$$(a\ i_1 \cdots i_r\ b\ j_1 \cdots j_s)(a\ b) = (a\ j_1\ j_2 \cdots j_s)(b\ i_1 \cdots i_r) \qquad (*)$$

$a \mapsto b \mapsto j_1$

$b \mapsto a \mapsto i_1$

$$f((a\ i_1 \cdots i_r\ b\ j_1 \cdots j_s)(a\ b)) = s + r$$

$$f((a\ i_1 \cdots i_r\ b\ j_1 \cdots j_s)) = s + r + 1$$

$$(a\ j_1 \cdots j_s)(b\ i_1 \cdots i_r)(a\ b) = (a\ i_1 \cdots i_r\ b\ j_1 \cdots j_s) \qquad \blacksquare$$

Def The alternating group A_n is the kernel of $\sigma : S_n \longrightarrow \mathbb{Z}/2$.

Remark A group is simple if it has no nontrivial proper normal s.g. $\mathbb{Z}/p$ is simple for p prime. A_n is simple for $n \gg 0$.

$\sigma : S_n \longrightarrow \mathbb{Z}/2$ is surjective

$(1\ 2) \longmapsto -1$

$$\Rightarrow |A_n| = \frac{n!}{2} = |S_n| / |\mathbb{Z}/2|$$

$i \mapsto j$ $\quad k \mapsto l$
$j \mapsto i$ $\quad l \mapsto k$

Prop 39 A_n is generated by 3-cycles.

Pf: A_n is generated by $(i\,j)(k\,l) = (i\,k\,j)(i\,k\,l)$

$$(i\,j)(j\,k) = (k\,i\,j)$$ $\blacksquare$

Conjugacy in S_n

"Def" The conjugacy class of $a \in G$ is the equivalence class of a under the relation $a \sim b$ iff $a = gbg^{-1}$ some g.

Prop 40 If $b = (i_1 \cdots i_r)$, then for any $a \in S_n$

$$aba^{-1} = (a(i_1) \ldots a(i_r)).$$

Pf: $j \notin \{a(i_1), \ldots, a(i_r)\} \Rightarrow a^{-1}(j) \notin \{i_1, \ldots, i_r\}$

$\Rightarrow b(a^{-1}(j)) = a^{-1}(j). \Rightarrow \underset{(aba^{-1})(j)}{\underset{\parallel}{a(b(a^{-1}(j)))}} = j$

$$\underbrace{a(i_k) \xmapsto{a^{-1}} i_k \xmapsto{b} i_{k+1} \xmapsto{a}}_{(aba^{-1})} a(i_{k+1})$$

$\blacksquare$

Cor 41 a) The conjugacy class of $b \in S_n$ is determined by its decomp into disjoint cycles.

b) Every cycle of length r is conj to $(1 \ldots r)$.

Pf: a) follows from b) because conj is a homomorphism and we can choose our conj elements to be disjoint.

b) $b = (i_1 \cdots i_r)$ $\qquad a(i_1) = 1,\ a(i_2) = 2, \ldots, a(i_r) = r$

Pick anything for remaining values.

$(i_{1,1} \cdots i_{r_1,1})(i_{2,1} \cdots i_{r_2,2}) \cdots \qquad (1 \cdots r_1)(r_1+1 \cdots r_1+r_2) \cdots$

Conjugate by
$i_{1,1} \longmapsto 1$
$i_{2,1} \longmapsto 2$
$\vdots$
$i_{r,1} \longmapsto r$
$i_{2,1} \longmapsto r+1$
$\vdots$

$\blacksquare$

$b = (1\ 3\ 7)(2\ 4\ 6\ 8)(5)$

$$a = \begin{pmatrix} 1 & 3 & 7 & 2 & 4 & 6 & 8 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

$$aba^{-1} = \underbrace{a(1\ 3\ 7)a^{-1}}_{(1\ 2\ 3)} \underset{?}{\cdot} \underbrace{a(2\ 4\ 6\ 8)a^{-1}}_{(4\ 5\ 6\ 7)} \cdot \underbrace{a(5)a^{-1}}_{(8)} = (4567)(123)(8)$$

$$\begin{pmatrix} 4 & 5 & 6 & 7 & 1 & 2 & 3 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

$(1\ 2\ 3\ 4)(5\ 6\ 7)(8)$

Rings

Def A ring is a set R together with 2 binary operations $+, \cdot$ s.t.

1) $(R,+)$ is an abelian group
2) $a\cdot(b\cdot c) = (a\cdot b)\cdot c$
3) $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$

   $(a+b)\cdot c = (a\cdot c)+(b\cdot c)$

$(R,+)$ is the underlying group.

$\underline{\text{Rng}}$ = category of rings

$$U: \underline{\text{Rng}} \longrightarrow \underline{\text{AbGrp}}$$
$$(R,+,\cdot) \longmapsto (R,+)$$

Def A non-zero ring $(R \neq \{0\})$ is unital if there is an element $1 \in R$ s.t. $1\cdot r = r\cdot 1 = r$ for all $r \in R$.

A ring is commutative if $a\cdot b = b\cdot a$ for all $a,b \in R$