Introduction:

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OH: TBA (please email conflicts)

Syllabus and course expectations on the web.

Grades: 1) Regular (weekly or semi-weekly) HW
2) Maybe a final project
No Mid-terms, no final

Course Outline:  
Book: Adkins & Weintraub, GTM #134

Goal: Structure theory of modules over a PID

Set-up: tease apart what we know for v. spaces

Field: Set F w/ 2 operations: +, · , s.t.
1) a + (b + c) = (a + b) + c
2) ∃ e s.t. a + e = e + a = a
3) ∀ a, ∃ a¹ s.t. a + a¹ = a¹ + a = e
4) a + b = b + a
5) a · (b · c) = (a · b) · c
6) a · (b + c) = (a · b) + (a · c)
7) (a + b) · c = (a · c) + (b · c)
8) ∃ 1 s.t. a · 1 = 1 · a = a
9) a · b = b · a
10) ∀ a ≠ 0, ∃ a¹ s.t. a · a¹ = a¹ · a = 1

Also study "vector spaces" over a ring R: R-modules
(~ last 1/3 of course)
Today: Set theory prelims \& relations/ equiv relations
For us: a set is a collection of objects called elements.
Very simple \& nothing super fancy.
Two big operations: such that \( x \) is an element of
\[
\text{Union } A \cup B = \{ x \mid x \in A \text{ or } x \in B \}
\]
\[
\text{Intersection } A \cap B = \{ x \mid x \in A \text{ and } x \in B \}
\]
If every element of \( A \) is an element of \( B \), then \( A \) is a subset of \( B \): \( A \subseteq B \)
Have some natural relations

Prop 1) \( A \subseteq B \iff A \subseteq A \cup B \)

This is an instance of a "universal property." Most of the objects we will study fit into "categories" (more on this later) and universal properties provide nice, intrinsic definitions.

Recall: 2) If \( C \) is such that \( C \subseteq B \) \& \( C \subseteq A \), then \( C \subseteq A \cap B \), i.e.

\[
C \subseteq A \cap B \iff \forall x \in A \cap B, x \in C
\]

3) Union is "dual":

If \( A \subseteq C \), \( B \subseteq C \), then \( A \cup B \subseteq C \).

Complement: \( B - A = \{ x \mid x \in B \text{ and } x \notin A \} \)

Two sets \( A \not\subseteq B \) are equal if \( A \subseteq B \) and \( B \subseteq A \).
\( \iff \) have the same elements.
Some important sets:

Name          Notation  \(\text{Elements/set-builder}\)
Empty Set     \(\emptyset\)  \(\{3\}\)
Natural Numbers \(\mathbb{N}\)  \(\{0, 1, \ldots \}\)
Integers      \(\mathbb{Z}\)  \(\{\ldots, -1, 0, 1, \ldots \}\)
Rationals     \(\mathbb{Q}\)  \(\{\frac{a}{b} \mid a, b \in \mathbb{Z}, \text{ no common factors}\}\)
Reals         \(\mathbb{R}\)  \(\)
Def: A relation from A to B is a subset R of A x B.

Normally write $arb$ for "$(a, b) \in R$" ie "a is related to b"

A function is a special type of relation:

A relation is a function if:
1) $\forall a \in A, \exists b \text{ s.t. } arb$
2) If $arb \neq arc$, then $b = c$

In words "arb means $b = f(a)$". The subset of $A \times B$ is the graph of $f$ and 2) is the "vertical line test".

Def: Let R be a relation on A. Then:
1) R is reflexive if $\forall a \in A$, $ara$
2) R is symmetric if $arb \Rightarrow bra$
3) R is transitive if $arb, brc \Rightarrow arc$
4) R is ordering if $arb, bra \Rightarrow a = b$

Def: R is an equivalence relation if 1), 2), 3) hold.

b) R is a partial order if 1), 3), 4) hold.

Ex: A is any set, $arb$ iff $a = b$ (R is the relation of "equality")
this is the prototype of an equivalence relation.

Ex: $A = \mathbb{Z}$, $arb$ R $a \geq b$
this is the prototype of a partial order.

Ex: $A = \mathbb{Z}$, $a \sim b$ if $a - b$ is divisible by $n$.
this is an equivalence relation.

Def: If A is a set and R on A is an equivalence relation,
then the equivalence class of a is
$[a] = \{b | arb \}$.

Since R is an equivalence relation:
Lemma 1. If $a, b \in A$, then either $[a] \cap [b] = \emptyset$ or $[a] = [b]$.

Proof If $[a] \cap [b] = \emptyset$, done.

Otherwise, let $c \in [a] \cap [b]$. Then $aRc \land bRc \Rightarrow aRb$ or $bRb \Rightarrow a \in [b], b \in [a]$

Now if $d \in [b]$, then $dRa, bRa \Rightarrow dRa \Rightarrow d \in [a]$. In other words $[b] \subseteq [a]$. Symmetry gives the rest. $\square$

Definition 1. If $A$ is a set, $R$ an equivalence on $A$, then the quotient of $A$ by $R$ is the set of equivalence classes:

$$A/R = \{ [a] \mid a \in A \}$$

read "$A$ mod $R$"

There is a natural function

$$\pi : A \rightarrow A/R$$

$$a \mapsto [a]$$

This is universal for maps respecting the equivalence relation (ie $aRb \Rightarrow \pi(a) = \pi(b)$).

Example 1. For $\mathbb{Z}$, $\sim$, have $n$ equivalence classes:

$[0], [1], ..., [n-1]$

corresponding to the remainder of division by $n$. 