Solution Set #2

Note Title 2/20/2008

30a: If $H \triangleleft G$, $[G:H]=n$, then G/H is a group of order n. $\Rightarrow \forall a \in G$, $o([a])$ divides $n = |G/H| \Rightarrow [a]^n = [e] \longleftrightarrow a^n \in H$.

30b: Let $G = S_3$, $H = \{e, (1\,2)\} \cong \mathbb{Z}/2$. Then $[G:H]=3$, but $(23)^3 = (23) \notin H$.

32a: The class equation says $|G| = |Z(G)| + \sum_i [G:\mathrm{Stab}_G(a)]$, where a ranges over the conjugacy classes in G with more than 1 element. $\Rightarrow [G:\mathrm{Stab}_G(a)] = p^k$ some $k>0$ $\Rightarrow 0 = |Z(G)| \mod p$. Since $e \in Z(G)$, $|Z(G)| \neq 0 \Rightarrow |Z(G)|$ is a non-trivial subgroup (a p-group, actually).

32b: By induction on n, $|G| = p^n$. By 32a, $|Z(G)|>1$ and $0 \mod p \Rightarrow$ have a subgroup $H \subseteq Z(G)$ of order p. This subgroup is normal, so G/H is a group of order p^{n-1}. $\Rightarrow$ has subgroups of order p^j $0 \le j \le n-1$ & correspondence theorem gives the result.

34a: $\mathbb{Z}$ is the free group on one generator $\Rightarrow \mathrm{Hom}_{Grp}(\mathbb{Z}, \mathbb{Z}_n) = \mathrm{Hom}_{Set}(\{1\}, o(\mathbb{Z}_n)) = \mathbb{Z}_n$.

34b: Let g generate $\mathbb{Z}/7$. Then for all $\phi: \mathbb{Z}/7 \to \mathbb{Z}/16$, $\phi(g)$ has order dividing 7. In $\mathbb{Z}/16$, the elements have order $1, 2, 4, 8, 16$, so $\phi(g)$ must have order $1 \Rightarrow \phi(g) = 0 \in \mathbb{Z}/16$. $\Rightarrow \mathrm{Hom}(\mathbb{Z}/7, \mathbb{Z}/16) = 0$.

35: Let $\phi: \mathbb{Z}/n \to \mathbb{Z}/m$. Since $\mathrm{Im}(\phi) \subseteq \mathbb{Z}/m$, $|\mathrm{Im}(\phi)| \mid m$. Since $\mathrm{Im}(\phi) \cong (\mathbb{Z}/n)/\ker\phi$, $|\mathrm{Im}(\phi)| \mid n \Rightarrow |\mathrm{Im}(\phi)| \mid d = \gcd(m,n) \Rightarrow \mathrm{Im}(\phi) \subseteq \left(\frac{m}{d}\right)\mathbb{Z}/m \cong \mathbb{Z}/d$. Since a homomorphism $\mathbb{Z}/n \to \mathbb{Z}/m$ is determined by where it sends $1 \in \mathbb{Z}/n$, we need only show that for all $r\left(\frac{m}{d}\right) \in \left(\frac{m}{d}\right)\mathbb{Z}/m$, the map $1 \mapsto \frac{rm}{d}$ gives a homomorphism. Thus we must check that $n\left(\frac{rm}{d}\right) \equiv 0 \mod m$. Since $d \mid n$, $n\left(\frac{rm}{d}\right) = \left(\frac{n}{d}\right) rm$, so we are done.

In fact, $\mathrm{Hom}(A,B)$ is an abelian group if A & B are. The argument given shows that $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/d$, with a nice generator determined by $1 \mapsto \left(\frac{m}{d}\right)$.

40a:

- $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix} = (1\ 6\ 3\ 4)(2\ 5)$

- $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 3 & 6 & 5 & 7 & 4 & 2 \end{pmatrix} = (1\ 8\ 2)(4\ 6\ 7)$

- $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \end{pmatrix} = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$

- $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 8 & 9 & 2 & 1 & 4 & 3 & 6 & 7 \end{pmatrix} = (1\ 5)(2\ 8\ 6\ 4)(3\ 9\ 7)$

47a: $D_8 = \{x, y \mid x^4 = y^2 = 1,\ yxy = x^3\}$

order:	1	2	4	8
s.gps: (normal)	$\{e\}$	$\langle x^2 \rangle$ ← center	$\langle x \rangle$	D_8
		$\langle y \rangle$	$\{x^2, y, xy, e\}$	
		$\langle x^2 y \rangle$	$\{e, x^2, xy, x^3 y\}$	
		$\langle xy \rangle$		
		$\langle x^3 y \rangle$		

47b. $Q_8 = \{\pm 1, \pm i, \pm j, \pm k \mid i^2 = j^2 = k^2 = -1,\ ij = k,\ (jk = i,\ ki = j)\}$

Only 1 element of order 2: -1. All other elements are order 1 or 4. $\Rightarrow D_8 \not\cong Q_8$.

49. $312 = 24 \cdot 13 = 2^3 \cdot 3 \cdot 13$

So if $n = \#$ 13 sylow s.g, then $n \equiv 1 \bmod 13$, $n \mid 2^3 \cdot 3$

Factors of 24:	Mod 13
24	11
12	12
8	8
6	6
4	4
3	3
2	2
1	1

So there is 1 13-Sylow s.g $\Rightarrow$ is normal.

50. $56 = 2^3 \cdot 7$

If $n = \#$ 7 sylow, then $n \equiv 1 \bmod 7$, $n \mid 8$

$n=1$, done. $n=8$, then there are 48 elements of order exactly 7 (6 for each 7 sylow s.g). $\Rightarrow$ 8 elements of order dividing 8 $\Rightarrow$ these elements are the 2-sylow s.g & any conjugate lies in these same elements. $\Rightarrow$ 2-Sylow is normal.

52. $168 = 2^3 \cdot 3 \cdot 7$

Let $n = \#7$Sylow s.g. Since 7 divides 168 but 49 doesn't, these s.g are all $\mathbb{Z}/7$ and every element of order 7 is inside one. $n \equiv 1$ (7) and $n | 24$

factors :	24	12	8	6	4	2	1
mod 7 :	3	5	1	6	4	2	1

The group is simple $\Rightarrow n > 1 \Rightarrow n = 8. \Rightarrow$ 48 elements of order 7 (6 for each Sylow s.g).

These problems are optional

↓

48. $(\mathbb{Z}/p^2)^* \cong \mathbb{Z}/p(p-1) \cong \mathbb{Z}/p \times \mathbb{Z}/p-1$. We'll look at $\mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$. (If $p=2$, D_8 & Q_8 work. Assume $p>2$)

There are p maps $\mathbb{Z}/p \longrightarrow \mathbb{Z}/p \times \mathbb{Z}/p-1$
$g \longmapsto (g^k, 0)$ $\quad 0 \le k \le p-1$

Choosing $k=0$ gives the direct product. Try $k=1$ (all choices of $k \ne 0$ are isomorphic)

$$G_1 = \{ a, b \mid a^{p^2} = 1,\ b^p = 1,\ bab^{-1} = a^{1+p} \}$$

Note that this has an element of order p^2.

Now look at $(\mathbb{Z}/p)^2$. $Aut(\mathbb{Z}/p \times \mathbb{Z}/p) = GL_2(\mathbb{F}_p)$ has order $(p^2-1)(p^2-p) = p(p^2-1)(p-1)$. There is a subgroup of order p (given by $\begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$, $0 \le k \le p-1$), so we have a non-trivial map $\mathbb{Z}/p \to Aut(\mathbb{Z}/p^2)$.
Let $G_2 = (\mathbb{Z}/p)^2 \rtimes_\phi \mathbb{Z}/p$ where ϕ is this map. Then every element in G_2 has order p:
Let x, y generate $(\mathbb{Z}/p)^2$, and let z generate $\mathbb{Z}/p$. Then y generates a subgroup of the center (in fact the whole center) & $zxz^{-1} = xy^{-1}$ or $xz = zxy$. Thus $(x^i y^j z^k)^p =$
$(x^i z^k)^p = x^{ip} z^{kp} y^{i \cdot k(1+2+\dots+(p-1))} = \{e\}$. This was trickier than I thought!
↑ expand this out and commute past.

$$G_2 \cong \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right\} \subseteq SL_3(\mathbb{Z}/p)$$ is the Heisenberg group of order p^3.

53. Sylow Thm ⇒ 3-Sylow s.g H is normal. If K is a 2-Sylow s.g, then $H \cap K = \{e\}$.
⇒ $|G| = 18$, then $G \cong H \rtimes K$. If K normal, then $G = H \times K$, so 2 groups

$G = \mathbb{Z}/9 \times \mathbb{Z}/2$ $(\mathbb{Z}/18)$ and $\mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}/2$ $(\mathbb{Z}/3 \times \mathbb{Z}/6)$

Assume K not normal. Then G is determined by the map $\mathbb{Z}/2 \longrightarrow Aut(H)$.

2 cases:

1) $H = \mathbb{Z}/9$ ⇒ $Aut(H) = \mathbb{Z}/6$ & only one non-trivial homomorphism: $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/6$, $1 \longmapsto 3$
So we get D_{18}.

2) $H = \mathbb{Z}/3 \times \mathbb{Z}/3$ ⇒ $Aut(H) = GL_2(\mathbb{F}_3)$. $|GL_2(\mathbb{F}_3)| = 2^4 \cdot 3$, so we have (maybe lots of) subgroups of order 2. To get non-isomorphic groups, we have to take 2 homomorphisms $\mathbb{Z}/2 \longrightarrow GL_2\mathbb{F}_3$ that are not conjugate: The homomorphisms are determined by their

values on $1 \in \mathbb{Z}/_2 = \{0,1\}$, and we want those s.t. the matrices are not similar. A "fun" exercise shows there are 2: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (Jordan normal form). $\Rightarrow$ 2 groups

$G_{18} = (\mathbb{Z}/_3)^2 \rtimes \mathbb{Z}/2$ by $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and

$S_3 \times \mathbb{Z}/_3 = (\mathbb{Z}/_3)^2 \rtimes \mathbb{Z}/_2$ by $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.