Solution Set #2

30a: If $H \triangleleft G$, $[G:H]=n$, then $G/H$ is a group of order $n. \Rightarrow \forall a \in G$, $o([a])$ divides $n \Rightarrow [a]^n = [e] \iff a^n \in H$.

30b: Let $G=S_3$, $H = \{ e, (1 2) \} \cong \mathbb{Z}/2$. Then $[G:H]=3$, but $(23)^3 = (23) \notin H$.

32a: The class equation says $|G| = |Z(G)| + \sum [G:Stab_G(a)]$, where $a$ ranges over the conjugacy classes in $G$ with more than 1 element. $\Rightarrow [G:Stab_G(a)] = p^k$ some $k > 0 \Rightarrow 0 = |Z(G)| \mod p$. Since $e \in Z(G)$, $|Z(G)| \neq 0 \Rightarrow |Z(G)|$ is a non-trivial subgroup (a $p$-group, actually).

32b: By induction on $n$, $|G| = p^n$. By 32a, $|Z(G)| > 1$ and $0 \mod p \Rightarrow$ have a subgroup $H \leq Z(G)$ of order $p$. This subgroup is normal, so $G/H$ is a group of order $p^{n-1}$. $\Rightarrow$ has subgroups of order $p^j$ for $0 \leq j \leq n - 1$ \cite{correspondence theorem}. Gives the result.

34a: $M_3$ is the free group on one generator $\Rightarrow \text{Hom}(M_3; \mathbb{Z}/2) = \text{Hom}(\{e, s \}; \mathbb{Z}/2) = \mathbb{Z}/2$.

34b: Let $g$ generate $\mathbb{Z}/7$. Then for all $\phi: \mathbb{Z}/7 \to \mathbb{Z}/16$, $\phi(g)$ has order dividing 7. In $\mathbb{Z}/16$, the elements have order 1, 2, 4, 8, 16, so $\phi(g)$ must have order 1 $\Rightarrow \phi(g) = 0 \in \mathbb{Z}/16$.

$\Rightarrow \text{Hom}(\mathbb{Z}/7; \mathbb{Z}/16) = 0$.

35: Let $\phi: \mathbb{Z}/n \to \mathbb{Z}/m$. Since $\text{Im}(\phi) \leq \mathbb{Z}/m$, $|\text{Im}(\phi)| \mid m$. Since $\text{Im}(\phi) \cong (\mathbb{Z}/n)/\ker \phi$.

$|\text{Im}(\phi)| \mid n \Rightarrow |\text{Im}(\phi)| \mid d = \gcd(m, n) \Rightarrow \text{Im}(\phi) \cong (\mathbb{Z}/d) \leq \mathbb{Z}/d$. Since a homomorphism $\mathbb{Z}/n \to \mathbb{Z}/m$ is determined by where it sends 1, let $n$, we need only show that for all $r \in (\mathbb{Z}/d)$, the map $1 \to r$ gives a homomorphism. Thus we must check that $n(rm) = 0 \mod m$. Since $\gcd(m, n) = \gcd(d, n)$, so we are done.

In fact, $\text{Hom}(\mathbb{Z}/n; \mathbb{Z}/m) \cong \mathbb{Z}/d$, with a nice generator determined by $1 \to (\mathbb{Z}/d)$.

$40a: \cdot \begin{pmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 4 & 2 & 5 \end{pmatrix}$

$\cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 3 & 5 & 7 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 8 & 2 \\ 4 & 6 & 7 \end{pmatrix}$
47a. \( D_8 = \langle x, y | x^4 = y^2 = 1, yxy = x^3 \rangle \)

Order: \(1, 2, 4, 8\)

S.GPs: \( \{e, x^2, x, xy, x^3y \} \)

(normal) \( \langle y \rangle, \langle x^2 \rangle, \langle x \rangle \)

47b. \( Q_8 = \langle 1, i, j, k | i^2 = j^2 = k^2 = -1, ij = k, (ji = i, ki = j)^2 \rangle \)

Only 1 element of order 2: -1. All other elements are order 1 or 4. \( \Rightarrow D_8 \not\cong Q_8. \)

48. \( 3!2 = 24 \cdot 13 = 2^3 \cdot 3 \cdot 13 \)

So if \( n = 13 \) Sylow S.G., then \( n = 1 \mod 13, n \mid 2^3 \cdot 3 \)

Factors of 24: \( \mod 13 \) So there is 1 13-Sylow S.G.

\[
\begin{array}{cccc}
24 & 11 & \text{is normal} \\
12 & 12 \\
8 & 8 \\
6 & 6 \\
4 & 4 \\
3 & 3 \\
2 & 2 \\
1 & 1 \\
\end{array}
\]

50. \( 56 = 2^3 \cdot 7 \)

If \( n = 7 \) Sylow, then \( n = 1 \mod 7, n \mid 8 \)

\( n = 1 \) done. \( n = 8, \) the there are 48 elements of order exactly 7 (6 for each 7 Sylow S.G.). \( \Rightarrow \) 8 elements of order dividing 8 \( \Rightarrow \) these elements are the 2-Sylow S.G. Any conjugates lies in these same elements. \( \Rightarrow \) 2-Sylow is normal.
52. \(168 = 2^3 \cdot 3 \cdot 7\)

Let \(n = 27\) Sylow s.g. Since 7 divides 168 but 49 doesn't, these Sylow s.g are all \(\mathbb{Z}/7\) and every element of order 7 is inside one. \(n = 1 (7)\) and \(n \mid 24\)

Factors: 24 12 8 6 4 2 1

mod 7: 3 5 1 6 4 2 1

The group is simple \(\Rightarrow n > 1 \Rightarrow n = 8. \Rightarrow 48\) elements of order 7 (6 for each Sylow s.g.),

These problems are optional
48. \( (\mathbb{Z}/p^2)^* \cong \mathbb{Z}/p(\mathbb{Z}/p^2) \cong \mathbb{Z}/p \times \mathbb{Z}/p \). We'll look at \( \mathbb{Z}/p^2 \times \mathbb{Z}/p \). (If \( p \neq 2 \), \( D_8 \), \( Q_8 \) work.) Assume \( p \neq 2 \)

The operation maps \( \mathbb{Z}/p \to \mathbb{Z}/p \times \mathbb{Z}/p \)
\( g \to (g^k, 0) \quad 0 \leq k \leq p-1 \)

Choosing \( k = 0 \) gives the direct product. Try \( k = 1 \) (all choices of \( k \neq 0 \) are isomorphic.)

<table>
<thead>
<tr>
<th>( G )</th>
<th>( a, b )</th>
<th>( a^2 = 1, b^p = 1, bab^{-1} = a^{i+p^2} )</th>
</tr>
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Note that this has an element of order \( p^2 \).

Now look at \( (\mathbb{Z}/p)^2 \). \( \text{Aut}(\mathbb{Z}/p \times \mathbb{Z}/p) = \text{GL}_2(\mathbb{F}_p) \) has order \( (p^2-1)(p^2-p) = p(p-1)(p-1) \). There is a subgroup of order \( p \) (given by \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), \( 0 \leq k \leq p-1 \)), so we have a non-trivial map \( \mathbb{Z}/p \to \text{Aut}(\mathbb{Z}/p)^2 \).

Let \( G_2 = (\mathbb{Z}/p)^2 \times \mathbb{Z}/p \) where \( \mathbb{Z}/p \) is this map. Then every element in \( G_2 \) has order \( p \):

Let \( x, y \) generate \( (\mathbb{Z}/p)^2 \), and let \( z \) generate \( \mathbb{Z}/p \). Then \( y \) generates a subgroup of the center (in fact the whole center): \( xz \).\( z^y = xz \) or \( xz = x \cdot y \). Thus \( (x^i z^k)^p = (x^i z^k)^p \). Expand this out and commute past.

\( G_2 \cong \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \subseteq \text{SL}_3(\mathbb{Z}/p) \) is the Heisenberg group of order \( p^3 \).

53. Sylow Thm \( \Rightarrow \exists - \) Sylow S.G.H is normal. If \( K \) is a 2-Sylow S.G, then \( H \cap K = \{e, 3\} \).

\( |G| = 18 \), then \( G \cong H \times K \). If \( K \) normal, then \( G = H \times K \), so 2 groups:

\[ G = \mathbb{Z}/4 \times \mathbb{Z}/2 \quad \text{and} \quad \mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}/2 \]

Assume \( K \) not normal. Then \( G \) is determined by the map \( \mathbb{Z}/2 \to \text{Aut}(H) \).

2 cases:

1) \( H = \mathbb{Z}/4 \Rightarrow \text{Aut}(H) = \mathbb{Z}/4 \) ! only one non-trivial homomorphism: \( \mathbb{Z}/2 \to \mathbb{Z}/4 \)

So we get \( D_8 \).

2) \( H = \mathbb{Z}/3 \times \mathbb{Z}/3 \Rightarrow \text{Aut}(H) = \text{GL}_2(\mathbb{F}_3) \). \( |\text{GL}_2(\mathbb{F}_3)| = 2^4 \cdot 3 \), so we have (maybe lots of) subgroups of order 2. To get non-isomorphic groups, we have to take 2 homomorphisms \( \mathbb{Z}/2 \to \text{GL}_2(\mathbb{F}_3) \) that are not conjugate; the homomorphisms are determined by their
values on $1 \in \mathbb{Z}/2 = \{0, 1\}$, and we want those s.t. the matrices are not similar. A "fun" exercise shows there are $2$: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (Jordan normal form). $\Rightarrow$ 2 groups

$G_{18} = (\mathbb{Z}/3)^2 \times \mathbb{Z}/2$ by $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and

$S_3 \times \mathbb{Z}/3 = (\mathbb{Z}/3)^2 \times \mathbb{Z}/2$ by $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. 