Homework 1 Solutions

Note Title 2/5/2008

3. S_3 is generated by $\tau: \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ & $\sigma: \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$,

Check: $\sigma^3 = 1$, $\tau^2 = 1$,

$$\tau\sigma\tau = \underset{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}}{\underset{\shortparallel}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}}} \underset{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}}{\underset{\shortparallel}{\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}}}} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \sigma^2$$

$\Rightarrow \sigma\tau\sigma\tau = 1$

$\Rightarrow \sigma\tau\sigma = \tau$

$\Rightarrow \sigma\tau = \tau\sigma^2$

& $\sigma^2\tau\sigma^2 = \tau$

	1	σ	σ^2	τ	$\tau\sigma$	$\tau\sigma^2$
1	1	σ	σ^2	τ	$\tau\sigma$	$\tau\sigma^2$
σ	σ	σ^2	1	$\tau\sigma^2$	τ	$\tau\sigma$
σ^2	σ^2	1	σ	$\tau\sigma$	$\tau\sigma^2$	τ
τ	τ	$\tau\sigma$	$\tau\sigma^2$	1	σ	σ^2
$\tau\sigma$	$\tau\sigma$	$\tau\sigma^2$	τ	σ^2	1	σ
$\tau\sigma^2$	$\tau\sigma^2$	τ	$\tau\sigma$	σ	σ^2	1

13: a) G is abelian iff $g \overset{f}{\longmapsto} g^{-1}$ is a gp hom.

Pf: $f(gh) = (gh)^{-1} = h^{-1}g^{-1}$ while

$f(g)f(h) = g^{-1}h^{-1}$.

So G is abelian iff $h^{-1}g^{-1} = g^{-1}h^{-1}$ iff $f(gh) = f(g)f(h)$ $\forall g,h \in G$

(every element is the inverse of some other element).

b) G is abelian iff $g \overset{f}{\longmapsto} g^2$ is a gp hom:

$f(gh) = (gh)(gh)$

$f(g)f(h) = gghh$

So $f(gh) = f(g)f(h) \longleftrightarrow ghgh = gghh \longleftrightarrow hg = gh \leftrightarrow G$ ab.

18. The argument that $[G:K] = [G:H][H:K]$ works to show that if the RHS is finite, then the LHS is finite. So it suffices to show that $[H:H\cap K] < \infty$.

This follows from the map $H/H\cap K \xrightarrow{\simeq} HK/K \subseteq G/K$. We can't use the 2nd isomorphism theorem, but we can use that the map $H/H\cap K \to HK/K$ defined by $h(H\cap K) \to hK$ is a bijection: $[h_1] \mapsto [\ell] \leftarrow [h_2]$, then $h_1^{-1}h_2 \in K \Rightarrow h_1^{-1}h_2 \in K\cap H. \Rightarrow$ 1-1. hK is the image of $h(H\cap K) \Rightarrow$ surjection. G/K is finite, so $|H/H\cap K| < \infty$.

20. 1) $Z(G)$ is abelian: $g, h \in Z(G)$. $\Rightarrow$ Since $g \in Z(G)$, $ga = ag$ $\forall a \in G$. In particular, for $a = h \Rightarrow Z(G)$ abelian.
 2) $Z(GL_n(\mathbb{R}))$: Test against elementary matrices.

21. Let $H \subseteq Z(G)$. Then $H \trianglelefteq G$.

Pf: Let $g \in G$, $h \in H$. Then $ghg^{-1} = (gh)g^{-1} \underset{\substack{\uparrow \\ h \in Z(G)}}{=} (hg)g^{-1} = h(gg^{-1}) = h$

So $ghg^{-1} = h \in H \Rightarrow H$ is normal.

22. a) Saw in class that $c(g)$ (conjugation by g) is a group hom. $\Rightarrow c(g)([a,b]) = [c(g)(a), c(g)(b)]$. Therefore conjugation by g takes generators of $[G,G]$ to other generators, so the image of conjugation by g on $[G,G]$ lies in $[G,G]$.

Now in G/G', $aG' \cdot bG' = bG' \cdot aG'$ since

$a^{-1}b^{-1}ab \in G' \Leftrightarrow (ba)^{-1}(ab) \in G' \Leftrightarrow [ab] = [ba]$.

b) Saw in class. If G/H is abelian, then $\ker(G \to G/H)$ contains G', but this kernel is H.

25. If G has exactly one s.g H of order n, then $H \trianglelefteq G$.

Pf: Since $c(g)$ is a group hom, $c(g)(H)$ is a subgroup of G. Since $c(g)$ is a bijection, $|c(g)(H)| = n$, so by the uniqueness assumption, $c(g)(H) = gHg^{-1} = H \Rightarrow H$ is normal.

My Problem: Define $H(d) = (h(d), k(d))$. A priori, this is in $A \times B$, but since $f(h(d)) = g(k(d))$, $H(d) \in A \times_C B$. This gives existence. For uniqueness, if $H': D \to A \times_C B$ is s.t. $\pi_A \circ H' = h$, $\pi_B \circ H' = k$, then since $A \times_C B \subseteq A \times B$, we see immediately that $H'(d) = (h(d), k(d)) = H(d)$.