3. $S_3$ is generated by $\tau: \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ and $\sigma: \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

Check: $\sigma^3 = 1$, $\tau^2 = 1$,

\[
\tau \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \sigma^2
\]

\[
\tau \sigma \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \sigma^2
\]

\[
\sigma \tau \sigma \tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \sigma^2
\]

\[
\sigma^2 \tau \sigma^2 = \tau
\]

13. a) $G$ is abelian iff $g^f \rightarrow g^{-1}$ is a gp hom.

Pf: $f(gh) = (gh)^{-1} = h^{-1}g^{-1}$

\[
f(g)f(h) = g^{-1}h^{-1}
\]

So $G$ is abelian iff $h^{-1}g^{-1} = g^{-1}h^{-1}$ iff $f(gh) = f(g)f(h)$ for all $g, h \in G$

(every element is the inverse of some other element).

b) $G$ is abelian iff $g^f \rightarrow g^2$ is a gp hom:

\[
f(gh) = (gh)(gh)
\]

\[
f(g)f(h) = gg hh
\]

So $f(gh) = f(g)f(h)$ iff $ghgh = gghh$ iff $hg = gh$ iff $G$ ab.

18. The argument that $[G:K] = [G:H][H:K]$ works to show that if the RHS is finite, then the LHS is finite. So it suffices to show that $[H:H \cap K] < \infty$. 


This follows from the map \( H/HnK \rightarrow HK/K \subset G/K \). We can't use the 2nd isomorphism theorem, but we can use that the map \( H/HnK \rightarrow HK/K \) defined by \( h(HnK) \rightarrow hK \) is a bijection: \( [h_1][h_2] \rightarrow [h_1h_2] \), i.e., \( h_1^{-1}h_2 \in K \Rightarrow h_1^{-1}h_2 \in KnH \Rightarrow 1-1 \). \( hK \) is the image of \( h(HnK) \Rightarrow \) surjection. \( G/K \) is finite, so \( |H/HnK| < \infty \).

20. 1) \( Z(G) \) is abelian: \( g, h \in Z(G) \Rightarrow \) Since \( g \in Z(G) \), \( ga = ag \ \forall a \in G \).

In particular, for \( a = h \Rightarrow Z(G) \) abelian.

2) \( Z(GL_n(R)) \): Test against elementary matrices.

21. Let \( H \in Z(G) \). Then \( H \triangleleft G \).

\[ \text{Pf:} \text{ Let } g \in G, \ h \in H. \text{ Then } \quad ghg^{-1} = (gh)g^{-1} = (hg)g^{-1} = h(gg^{-1}) = h \in Z(G) \]

So \( ghg^{-1} = h \in H \Rightarrow H \text{ is normal.} \)

22. a) Saw in class that \( c(g) \) (conjugation by \( g \)) is a group hom. \( \Rightarrow \) \( c(g)[[a, b]] = [c(g)(a), c(g)(b)] \). Therefore conjugation by \( g \) takes generators of \( [G, G] \) to other generators, so the image of conjugation by \( g \) on \( [G, G] \) lies in \( [G, G] \).

Now in \( G/G' \), \( aG' \cdot bG' = bG' \cdot aG' \text{ since } a^{-1}b^{-1}ab \in G' \Leftrightarrow (ba^{-1})(ab) \in G' \Leftrightarrow [ab] = [ba] \).

b) Saw in class. If \( G/H \) is abelian, then \( \bar{G} - (G \rightarrow G/H) \) contains \( G' \), but this kernel is \( H \).

25. If \( G \) has exactly one Sylow \( H \) of order \( n \), then \( H \triangleleft G \).

\[ \text{Pf:} \text{ Since } c(g) \text{ is a group hom, } c(g)(H) \text{ is a subgroup of } G. \]

Since \( c(g) \) is a bijection, \( |c(g)(H)| = n \), so by the uniqueness assumption, \( c(g)(H) = ghg^{-1} = H \Rightarrow H \text{ is normal.} \)

My Problem: Define \( H(d) = (h(d), h(\pi(d))) \). A priori, this is in \( A \times B \), but since \( f(h(\pi(d))) = g(h(\pi(d))) \), \( H(d) \in \text{A } \times \text{ B}. \) This gives existence. For uniqueness, if \( H' : D \rightarrow A \times B \) is s.t. \( \pi_A \circ H' = h, \pi_B \circ H' = \pi \), then since \( A \times B \in A \times B \), we see immediately that \( H'(d) = (h(d), h(\pi(d))) = H(d) \).