Fibonacci in Groups and Rings
University of Virginia
MATH 552 Intro to Abstract Algebra
Spring 08

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Fibonacci the Man

- Leonardo Pisano
- Lived from 1170 to 1250
- He was an Italian mathematician who has been called the most talented mathematician of the Middle Ages
- He is known for
  - Spreading the Hindu-Arabic numeral system in Europe
  - The Fibonacci numbers

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- $1/G(S) = 1 + G(S)$

Background

- Fibonacci groups were first introduced by J. H. Conway, the creator of the Game of Life (A cellular automaton)
- Used largely to test various computational techniques

Definitions

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- k(m) - the Fibonacci length of a sequence mod m

Definitions Cont.

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- Fibonacci Sequence - the first number of the sequence is 0, the second 1, and each subsequent number is equal to the sum of the previous two numbers
- Fibonacci Group - the Fibonacci group $F(n)$ is defined by $F(n) = < a_1, a_2, ..., a_n | a_i a_{i+1} = a_{i+2}, i = 1, ..., n >$ where the subscripts are reduced mod n to lie in the range 1,2,...n

Golden Matrix Ring

The matrix A =

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

can be used to derive an explicit formula for the Fibonacci Numbers in terms of the golden ratio, $\phi = (1+\sqrt{5})/2$, and its conjugate.

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The simplest ring generated by A is $\mathbb{Z}[A]$, the ring of polynomials in A with integer coefficients.

G.M.R. Cont.

In $\mathbb{Z}[A]$, $A^2 - A - 1 = 0$ because the characteristic polynomial of A is $det(XI - A) = X^2 - X - 1$, which is also the characteristic polynomial of the Fibonacci recurrence relation $F_{n+2} = F_{n+1} + F_n$.

G.M.R. Cont.

Since ϕ is the positive root of $\phi^2 - \phi - 1 = 0$, $\mathbb{Z}[A]$ and $\mathbb{Z}[\phi]$ are isomorphic under the eigenvalue map $\varepsilon : \mathbb{Z}[A] \to \mathbb{Z}[\phi]$ determined by $\varepsilon(A) = \phi$ and $\varepsilon(I) = 1$.

G.M.R. Cont.

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- $\mathbb{Z}[A] \cong \mathbb{Z}[\phi] = \mathbb{Z}[X]/(x^2 - x - 1)$
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- So $Ker =< x^2 - x - 1 >$

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- This means $\mathbb{Z}[\phi]$ is an integral domain.

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- Since ϕ is a unit, so is ϕ^n

G.M.R. Cont.

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- So the units are $\pm\phi^{\pm n}$

G.M.R. Cont.

By the isomorphism ε, $\mathbb{Z}[A]$ shares the same properties.

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$\mathbb{Z}[A]$ is called the Golden Matrix Ring

The Ring of Generalized Fibonacci Sequences

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- $\mathbb{F}$ is an abelian group under the addition $\{G_n\} + \{H_n\} = \{G_n + H_n\}$.

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- This gives you $G_n = b = \mathcal{G}(D)$.
- Define the sequence map $\mathcal{L} : \mathbb{Z}[A] \to \mathbb{F}$ by $\mathcal{L}(a + bA) = \{\mathcal{G}(A^n(a + bA))\}$.

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- Here $a = G_{n-1}$ and $b = G_n$
- This gives you $G_n = b = \mathcal{G}(D)$.
- Define the sequence map $\mathcal{L} : \mathbb{Z}[A] \to \mathbb{F}$ by $\mathcal{L}(a + bA) = \{\mathcal{G}(A^n(a + bA))\}$.
- Then $\mathcal{L}$ is a group homomorphism.

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- Thus, $\mathcal{L}$ and $\mathcal{M}$ form an inverse pair of group isomorphisms.
- We can transfer the multiplicative structure of $\mathbb{Z}[A]$ to $\mathbb{F}$ via $\mathcal{L}$ and $\mathcal{M}$.

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- With this multiplication, $\mathbb{F}$ becomes a ring, and the maps $\mathcal{M} : \mathbb{F} \to \mathbb{Z}[A]$ and $\mathcal{L} : \mathbb{Z}[A] \to \mathbb{F}$ are isomorphisms of rings.

Fibonacci Sequences and Groups

- An ordered pair (x_1, x_2) of elements of a group G determines a sequence in G by the rule $x_n x_{n+1} = x_{n+2}$, $n \in \mathbb{N}$.

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- When this sequence is periodic, its fundamental period is called the Fibonacci length of (x_1, x_2) in G.

An Example

For the group $\mathbb{Z}_2$ x $\mathbb{Z}_2$ = $< a, b | a^2$ = b^2 = $1 >$ we obtain the sequence

$$a, b, ab, bab, b, ba, a, b, \ldots$$

showing that the infinite dihedral group has Fibonacci length 6

A Second Example

The values of $U_n \pmod 7$ are

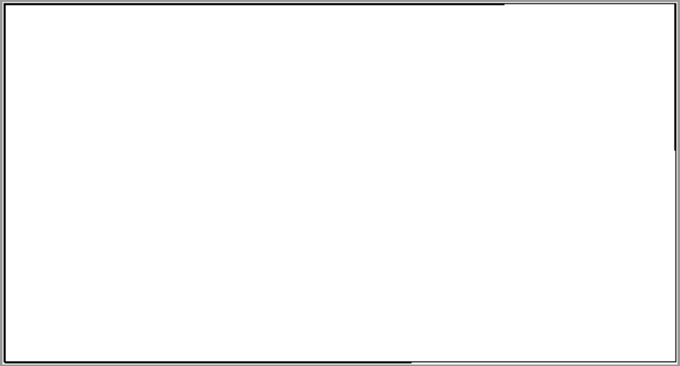
$$0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1$$

and then repeat; so $k(7) = 16$. Note that $U_8 = 0 \pmod 7$ so that the 16 terms in the period form two sets of 8 terms each.

Fibonacci in the $\mathbb{R}$ World

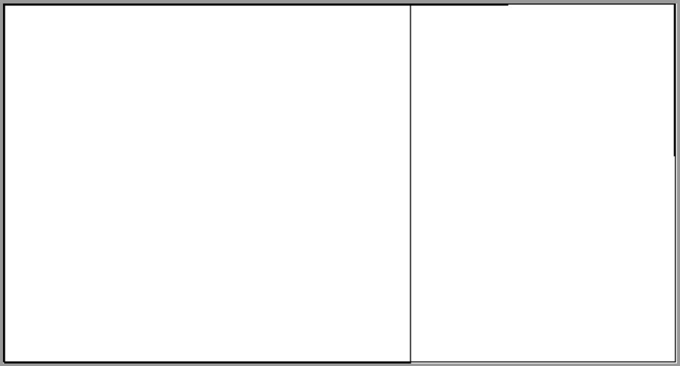
Fibonacci Spirals

Start with a rectangle



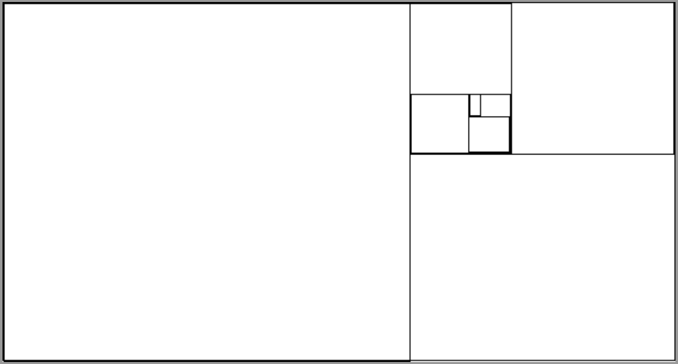
Spirals Cont.

Then you create a square and a rectangle within the original rectangle



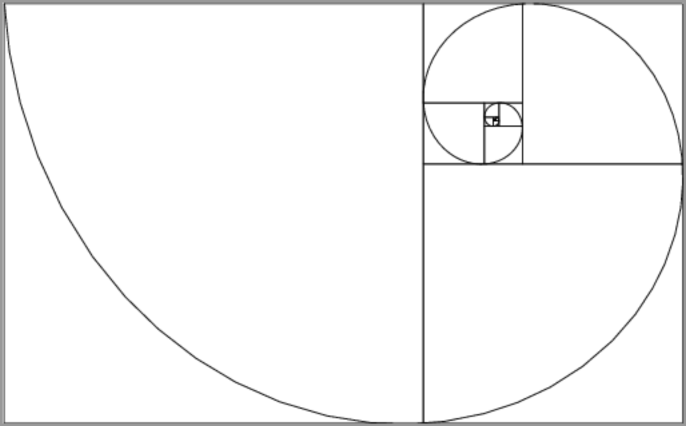
You continue doing this until you reach...

Spirals Cont.



Spirals Cont.

Then you draw a spiral within the rectangle...



Application to Nature

- The approach of a hawk to its prey: Their sharpest view is at an angle to their direction of flight; this angle is the same as the spiral's pitch.
- The arms of tropical storms
- The arms of spiral galaxies; the Milky Way, is believed to have 4 major spiral arms, each a Fibonacci spiral with pitch of about 12 degrees

Turku Power Station, Finland

Fibonacci Numbers in Music

Lateralus by Tool: If you count between pauses, the syllables in the verses from the first several Fibonacci numbers:

- (1) Black,
- (1) then,
- (2) white are,
- (3) all I see,
- (5) in my infancy,
- (8) red and yellow then came to be,
- (5) reaching out to me,
- (3) lets me see.
- (2) There is,
- (1) so,
- (1) much,
- (2) more that
- (3) beckons me,
- (5) to look through to these,
- (8) infinite possibilities.
- (13) As below so above and beyond I imagine,
- (8) drawn outside the lines of reason,
- (5) push the envelope,
- (3) watch it bend.

Music Cont.

The time signatures of the chorus change from 9/8 to 8/8 to 7/8, and the song's original name was 9-8-7. 987 is the 17th step of the Fibonacci sequence.