Lecture 8 - Partial Derivatives

Note Title

With more than one variable, we talk about partial derivatives instead of "the" derivative.

Partial derivatives measure the rate of change for each variable independently.

Def If $f(x,y)$ is a continuous function, then the partial derivative of f with respect to x at (a, b) is

$$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}.$$

Similarly, the partial derivative w.r.t y at (a,b) is

$$\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}.$$

To find $\frac{\partial}{\partial x}$, hold y constant & differentiate w.r.t. x as in 1-var calc.

To find $\frac{\partial}{\partial y}$, hold x constant & differentiate w.r.t. y as in 1-var calc.

Ex: $f(x,y) = x^3 + x^2y + xy^2 + y^3$

$f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = 3x^2 + 2xy + y^2$

$f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = x^2 + 2xy + 3y^2$

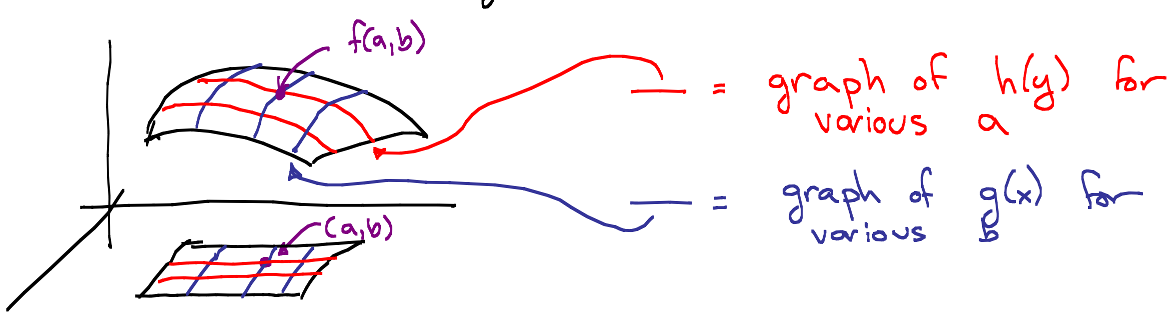
So $f_y(1,1) = 1^2 + 2\cdot1\cdot1 + 3\cdot1^2 = 6$

What does this measure? Rate of change in the corresponding direction:

If $g(x) = f(x,b)$ and $h(y) = f(a,y)$, then by def

$f_x(a,b) = g'(a)$ & $f_y(a,b) = h'(b)$.

Picture



So $f_y(a,b)$ is the slope of the tangent line to the red curve through (a,b), while $f_x(a,b)$ is the slope of the tangent line to the blue curve through (a,b).

Ex $f(x,y) = \cos(x^2+y^2)$

$$\frac{\partial f}{\partial x} = -2x\sin(x^2+y^2)$$

$$\frac{\partial f}{\partial y} = -2y\sin(x^2+y^2)$$

So at $(0,0)$, slope of the tangent line in the x-dir $= 0$;
slope of the tangent line in the y-dir $= 0$.

Say that $F(x,y,z)=0$ defines z implicitly if around every solution, we can solve for z.

Ex : $x^2+y^2+z^2-1=0$

$z = \sqrt{1-x^2-y^2}$
or
$z = -\sqrt{1-x^2-y^2}$

$x^2+y^2+z^3-1=0$

$z = \sqrt[3]{1-x^2-y^2}$

Just as in 1-var calc, we can find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

- Apply $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$ as usual, remembering that $\frac{\partial}{\partial x} z = \frac{\partial z}{\partial x}$ is a function of x and y.
- Solve for $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$

Ex: $x^2+y^2+z^2-1=0$ at $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

$\frac{\partial}{\partial x}$: $2x + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-x}{z} \rightsquigarrow -1$

$\frac{\partial}{\partial y}$: $2y + 2z\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{-y}{z} \rightsquigarrow -1$

Ex 2: $x^2+y^2+z^3-1=0$ @ $(1/\sqrt{2}, 0, 1/\sqrt[3]{2})$

$\frac{\partial}{\partial x}$: $2x + 3z^2\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-2x}{3z^2} \rightsquigarrow \frac{-\sqrt{2}}{3/\sqrt[3]{4}} = \frac{-2\sqrt[6]{2}}{3}$

$\frac{\partial}{\partial y}$: $2y + 3z^2\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{-2y}{3z^2} \rightsquigarrow 0$

Iterated partial derivatives

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are again functions of x & y. So we can form

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial x \partial x} f$$

Clairaut's Thm

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2}{\partial y \partial x} f \\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2}{\partial x \partial y} f \end{aligned}\right\}$$

If these are all continuous, then these two are equal

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2}{\partial y \partial y} f$$

Ex $f(x,y) = x^3 + x^2y + xy^2 + y^3$

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy + y^2$$

So

$$\frac{\partial f}{\partial y} = x^2 + 2xy + 3y^2$$

$$\frac{\partial^2 f}{\partial x \partial x} = 6x + 2y \qquad \frac{\partial^2 f}{\partial y \partial x} = 2x + 2y$$

$$\frac{\partial^2 f}{\partial y \partial y} = 6y + 2x \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x + 2y.$$

Ex: $f(x,y) = \cos(x^2+y^2)$

$$\frac{\partial f}{\partial x} = -2x\sin(x^2+y^2) \quad \xrightarrow{\partial/\partial x} \quad -4x^2\cos(x^2+y^2) - 2\sin(x^2+y^2)$$

$$\frac{\partial f}{\partial x} \xrightarrow{\partial/\partial y},\quad \frac{\partial f}{\partial y} \xrightarrow{\partial/\partial x} \quad -4xy\cos(x^2+y^2)$$

$$\frac{\partial f}{\partial y} = -2y\sin(x^2+y^2) \quad \xrightarrow{\partial/\partial y} \quad -4y^2\cos(x^2+y^2) - 2\sin(x^2+y^2)$$