With more than one variable, we talk about partial derivatives instead of "the" derivative.

Partial derivatives measure the rate of change for each variable independently.

**Def:** If \( f(x, y) \) is a continuous function, then the partial derivative of \( f \) with respect to \( x \) at \((a, b)\) is

\[
\frac{\partial f}{\partial x}(a, b) = f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}.
\]

Similarly, the partial derivative w.r.t. \( y \) at \((a, b)\) is

\[
\frac{\partial f}{\partial y}(a, b) = f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}.
\]

To find \( \frac{\partial}{\partial x} \), hold \( y \) constant & differentiate w.r.t. \( x \) as in 1-var. calc.

To find \( \frac{\partial}{\partial y} \), hold \( x \) constant & differentiate w.r.t. \( y \) as in 1-var. calc.

**Ex:** \( f(x, y) = x^3 + x^2y + xy^2 + y^3 \)

\[
f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = 3x^2 + 2xy + y^2
\]

\[
f_y(x, y) = \frac{\partial}{\partial y} f(x, y) = x^2 + 2xy + 3y^2
\]

So \( f_y(1, 1) = 1^2 + 2\cdot1\cdot1 + 3\cdot1^2 = 6 \)

What does this measure? Rate of change in the corresponding direction:

If \( g(x) = f(x, b) \) and \( h(y) = f(a, y) \), then by def

\[
f_x(a, b) = g'(a) \quad \text{and} \quad f_y(a, b) = h'(b).
\]

**Picture**

= graph of \( h(y) \) for various \( a \)

= graph of \( g(x) \) for various \( b \)
So \( f_y(a,b) \) is the slope of the tangent line to the red curve through \((a,b)\), while \( f_x(a,b) \) is the slope of the tangent line to the blue curve through \((a,b)\).

**Ex:**

\[
f(x,y) = \cos(x^2+y^2)
\]

\[
\frac{df}{dx} = -2x \sin(x^2+y^2)
\]

\[
\frac{df}{dy} = -2y \sin(x^2+y^2)
\]

So at \((0,0)\), slope of the tangent line in the \(x\)-dir = 0
slope of the tangent line in the \(y\)-dir = 0.

Say that \( F(x,y,z) = 0 \) defines \( z \) **implicitly** if around every solution, we can solve for \( z \).

**Ex:**

\[
x^2 + y^2 + z^2 - 1 = 0
\]

\[
z = \sqrt{1-x^2-y^2}
\]

\[
z = -\sqrt{1-x^2-y^2}
\]

Just as in 1-var calc, we can find \( \frac{dz}{dx} \) and \( \frac{dz}{dy} \)

1) Apply \( \frac{d}{dx} \) or \( \frac{d}{dy} \) as usual, remembering that \( \frac{d}{dx} z = \frac{dz}{dx} \) is a function of \( x \) and \( y \).

2) Solve for \( \frac{dz}{dx} \) or \( \frac{dz}{dy} \)

**Ex:**

\[
x^2 + y^2 + z^2 - 1 = 0
\]

\[
\frac{d}{dx} \quad 2x + 2z \frac{dz}{dx} = 0 \Rightarrow \frac{dz}{dx} = -\frac{x}{z} \quad \rightarrow \quad -1
\]

\[
\frac{d}{dy} \quad 2y + 2z \frac{dz}{dy} = 0 \Rightarrow \frac{dz}{dy} = -\frac{y}{z} \quad \rightarrow \quad -1
\]

**Ex 2:**

\[
x^2 + y^2 + z^2 - 1 = 0
\]

\[
\frac{d}{dx} \quad 2x + 3z^2 \frac{dz}{dx} = 0 \Rightarrow \frac{dz}{dx} = -\frac{2x}{3z^2} \quad \rightarrow \quad \frac{-\frac{\sqrt{3}}{3}}{\frac{\sqrt{14}}{3}} = -\frac{2\sqrt{2}}{3}
\]

\[
\frac{d}{dy} \quad 2y + 3z^2 \frac{dz}{dy} = 0 \Rightarrow \frac{dz}{dy} = -\frac{2y}{3z^2} \quad \rightarrow \quad 0
\]
Iterated partial derivatives

\[ \frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} \] are again functions of \( x \) \& \( y \). So we can form

\[ \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} \quad f \]

\[ \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x \partial y} \quad f \]

\[ \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} \quad f \]

\[ \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} \quad f \]

If these are all continuous, then

these two are equal

**Ex:** \( f(x, y) = x^3 + x^2 y + xy^2 + y^3 \)

\[ \frac{\partial f}{\partial x} = 3x^2 + 2xy + y^2 \]
\[ \frac{\partial f}{\partial y} = x^2 + 2xy + 3y^2 \]

\[ \frac{\partial^2 f}{\partial x \partial x} = 6x + 2y \]
\[ \frac{\partial^2 f}{\partial y \partial x} = 2x + 2y \]
\[ \frac{\partial^2 f}{\partial y \partial y} = 6y + 2x \]
\[ \frac{\partial^2 f}{\partial x \partial y} = 2x + 2y \]

**Ex:** \( f(x, y) = \cos(x^2 + y^2) \)

\[ \frac{\partial f}{\partial x} = -2x \sin(x^2 + y^2) \]
\[ \frac{\partial^2 f}{\partial x^2} = -4x^2 \cos(x^2 + y^2) - 2 \sin(x^2 + y^2) \]
\[ \frac{\partial f}{\partial y} = -2y \sin(x^2 + y^2) \]
\[ \frac{\partial^2 f}{\partial y^2} = -4y^2 \cos(x^2 + y^2) - 2 \sin(x^2 + y^2) \]