Lecture 4 – Surfaces

Note Title

Saw last time that the equation of the plane through (x_0, y_0, z_0) and $\perp$ to $\bar{n} = \langle a, b, c \rangle$ is $\bar{n} \cdot (\bar{r} - \bar{r}_0) = 0 \leftrightarrow \bar{n} \cdot \bar{r} = \bar{n} \cdot \bar{r}_0$

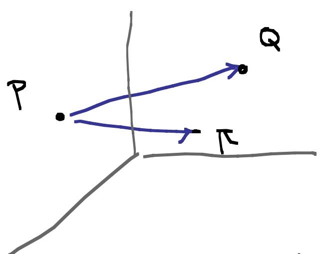
Expanded out: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Now let $d = ax_0 + by_0 + cz_0$. Then

$ax + by + cz = d$ ← standard form

Given 3 points, how can we find the plane containing them?

The vectors between the points all lie in the plane:



So choose one as $\bar{r}_0$ and let $\bar{n}$ be

$\overline{PQ} \times \overline{PR}$!

<u>Ex</u>: $\underset{P}{(1,0,0)}$, $\underset{Q}{(0,2,0)}$, $\underset{R}{(0,0,3)}$

$\overrightarrow{PQ} = \langle -1, 2, 0 \rangle$, $\overrightarrow{PR} = \langle -1, 0, 3 \rangle$

$$\Rightarrow \bar{n} = \overline{PQ} \times \overline{PR} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\bar{i} + 3\bar{j} + 2k$$

$\bar{r}_0 = \langle 1, 0, 0 \rangle$, so $d = \bar{n} \cdot \bar{r}_0 = 6$

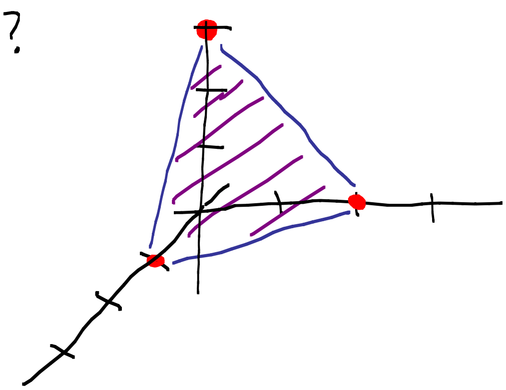
& plane is $\boxed{6x + 3y + 2z = 6}$

How do we sketch this?

Plot 3 points (usually where it hits axes)

Add in coord plane intercepts

Shade

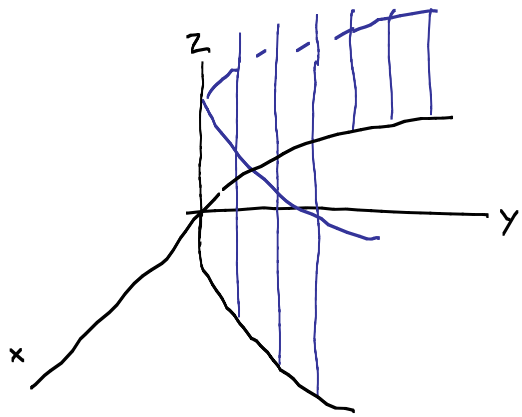


Today: Cylinders & Quadric Surfaces

<u>Def</u> A <u>cylinder</u> is the collection of all lines ∥ to a given line & passing through a fixed plane curve.

Basic cases: have a curve in some coord plane (say xy) and you let the other coord vary freely

<u>Ex</u> $y = x^2$ gives a cylinder. The lines are all ∥ to z-axis



How do we sketch it?

1. draw the curve
2. draw in the ∥ lines

What gives cylinders?

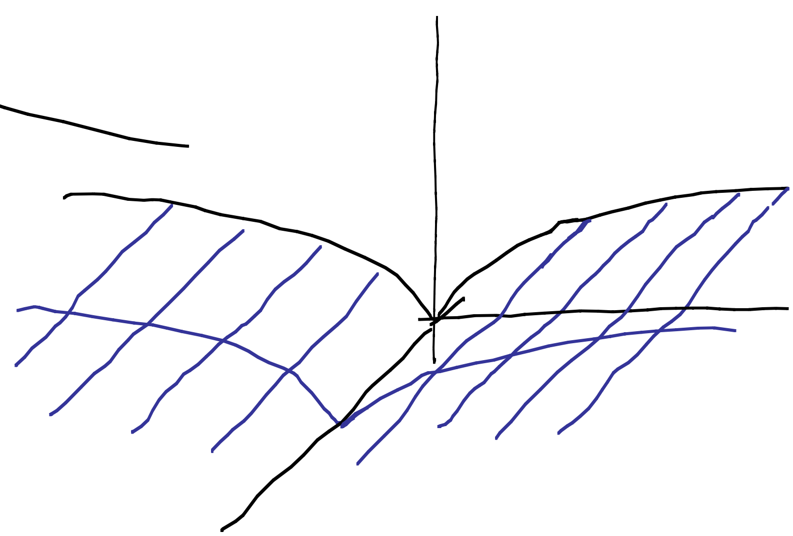
- any equation missing a variable
- any equation in which 2 var occur linearly

} big cases

<u>Ex</u>: $x = \sin z$

$y = z^{2/3}$

$x + y = z^4$



These are easy to sketch. Quadric surfaces are also easy:

Quadric surfaces are solutions to

$$Ax^2 + By^2 + Cz^2 + \cancel{Dxy} + \cancel{Exz} + \cancel{Fyz} + Gx + Hy + Iz + J = 0$$

we'll ignore these

3 cases: ① At least 2 of A, B, C are 0 ← cylinders!

② One of A, B, C is 0 } complete the square

③ None is 0

We will look at a few cases:

a) $z = ax^2 + by^2$ —— "paraboloids"

b) $ax^2 + by^2 + cz^2 = d$ —— "ellipses" and "hyperboloids"

How do we sketch? <u>Traces</u>

Set x, y, z in turn equal to various values. These give curves in the planes $x = k$ (or $y, z = k$).

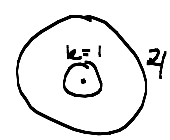
Ex: $z = x^2 + y^2$

$z = 0$: $x^2 + y^2 = 0 \Rightarrow x = y = 0$

$z = 1$: $x^2 + y^2 = 1 \Rightarrow$ unit circle

$\vdots$

$z = k$: $x^2 + y^2 = k \Rightarrow$ circles



x or y - traces:

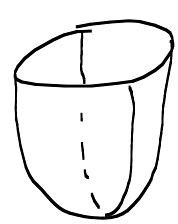
$x = 0$: $z = x^2 \Rightarrow$ parabola

$x = 1$: $z = x^2 + 1 \Rightarrow$ parabola

$\vdots$

$x = k$: $z = x^2 + k^2 \Rightarrow$ parabola

So surface is a paraboloid:

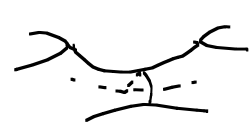


The others give shapes based on the coefficients:

$z = ax^2 + by^2$

$a, b > 0$
$a, b < 0$ $\Rightarrow$ paraboloid

$a < 0 < b$
$a > 0 > b$ $\Rightarrow$ hyperbolic paraboloid



$z^2 = ax^2 + by^2$ double cone

$ax^2 + by^2 + cz^2 = 1$

all positive $\Rightarrow$ ellipsoid

2 positive $\Rightarrow$ hyperboloid of 1 sheet

1 positive $\Rightarrow$ hyperboloid of 2 sheets

0 positive $\Rightarrow$ degenerate