Lecture 25 - Stokes' Theorem

Thm (Stokes' Theorem) Let \( S \) be an oriented surface with boundary curve \( C \). Then

\[
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}
\]

This has the same form as the F.T. of Line Ints: \( \int_C \nabla(f) \cdot dr \) on \( \mathbf{C} \) and add "eval \( f \) at \( \mathbf{C} \)"

Have to know how to orient \( C \) (otherwise the sign is wrong!) If \( S \) is an oriented surface, then \( C = \partial S \) inherits an orientation by requiring that the surface be on the left if we walk along the curve with our head in the direction of the normal vector \( \mathbf{N} \).

Make this precise by adding in the "outward pointing normal. Pick a point \( p \) on \( C \) and look at the tangent plane at \( p \). In this sits the line tangent to \( C \) and 2 unit vectors \( \mathbf{T} \). One points directly away from the surface (outward pointing normal), one points into the surface (inward pointing normal).
Then $C$ is oriented by taking $\mathbf{F}$ in the direction $\mathbf{N} \times$ (outward pt. norm)
if $S$ is the graph of a surface with the upward pointing normal, then $C$ is oriented counterclockwise when we look from above.

Ex: $C$ is the curve connecting $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$, oriented as $x$-axis $\rightarrow$ $y$-axis $\rightarrow$ $z$-axis.
Curve is on the plane $6x + 3y + 2z = 6$
or $\langle 3, \frac{3}{2}, 1 \rangle \cdot \langle x, y, z \rangle = 3$.

Let $\mathbf{F} = \langle e^{-x}, y, 3z, 2x + \sin (\cos z) \rangle$
Stokes' Theorem makes $\int_C \mathbf{F} \cdot d\mathbf{r}$ a breeze:

$$\nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
e^{-x} & y & 3z & 2x + \sin (\cos z)
\end{vmatrix} = \langle -3, -2, 1 \rangle$$

Surface: $6x + 3y + 2z = 6 \Rightarrow z = 3 - 3x - \frac{3}{2} y$

$\Rightarrow \quad \mathbf{n} = \langle -3, -\frac{3}{2}, 1 \rangle$ -- vector from above. For graph of $\alpha$ surface, $z$-comp always $\pm 1$!

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{2-2x} (-9 - 3 + 1) dy \, dx = \int_0^1 -22 + 22x \, dx = -22x + 11x^2 \bigg|_0^1 = -11.$$

Also makes surface integral easier:
Ex: $\mathbf{F} = \langle x e^z, y \cos z, z^2 y^3 \rangle$  $S =$ upper unit hemisphere, upward normal

$$\iint_S \nabla \times \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x e^z & y \cos z & z^2 y^3
\end{vmatrix} = \langle 32y^2 z^2 + y \sin z, x e^z - 2zxy^3, 0 \rangle$$
*S* is a part of a sphere, so must use spherical coords.

⇒ messy!

**Stokes:** \( \partial S = z=0 \) and \( x^2 + y^2 = 1 \).

Oriented so that unit circle is traversed counterclockwise.

\[
x = \cos t \\
y = \sin t \\
z = 0
\]

\( \Rightarrow \quad \bar{r}'(t) = \langle -\sin t, \cos t, 0 \rangle \)

\( \bar{F} = \langle x, y, 0 \rangle = \langle \cos t, \sin t, 0 \rangle \Rightarrow \bar{F} \cdot \bar{r}'(t) = -\cos t \sin t + \cos t \sin t = 0. \)

\( \Rightarrow \int_C \bar{F} \cdot d\bar{r} = \iiint_S \nabla \times \bar{F} \cdot d\bar{s} \)

**Last Point:** Stokes' theorem says that \( \iiint_S \nabla \times \bar{F} \cdot d\bar{s} \) is independent of the surface, covering only about \( \partial S \).

2 consequences:

1. If \( S \) is a closed surface, then \( \iiint_S \nabla \times \bar{F} \cdot d\bar{s} = 0 \)

2. If we can find \( S_2 \) s.t. \( \partial S_2 = \partial S \), then also get \( \iiint_{S_2} \nabla \times \bar{F} \cdot d\bar{s} = \iiint_S \nabla \times \bar{F} \cdot d\bar{s} \).

**Ex:** \( S: z = 1 - x^2 - y^2 \) \( \bar{G} = \langle -x, y, 2 \rangle \) \((= \nabla \times \langle y, x, -xy \rangle)\)

Tun

1. \( \bar{n} = \langle 2x, 2y, 1 \rangle \) \((S \text{ is a surface})\)

\( \Rightarrow \bar{G} \cdot \bar{n} = -2x^2 + 2y^2 + 2 \quad \text{can integrate.} \)

2. \( \partial S = \text{unit circle} \quad \text{or unit disk.} = \partial S_2 \)

on \( S_2: \bar{n} = \langle 0, 0, 1 \rangle \), so \( \bar{G} \cdot \bar{n} = 2 \)

\( \iiint_{S_2} \bar{G} \cdot \bar{n} \, dS = 2 \cdot \text{area (disk)} = \frac{2\pi}{3} \). Easy!