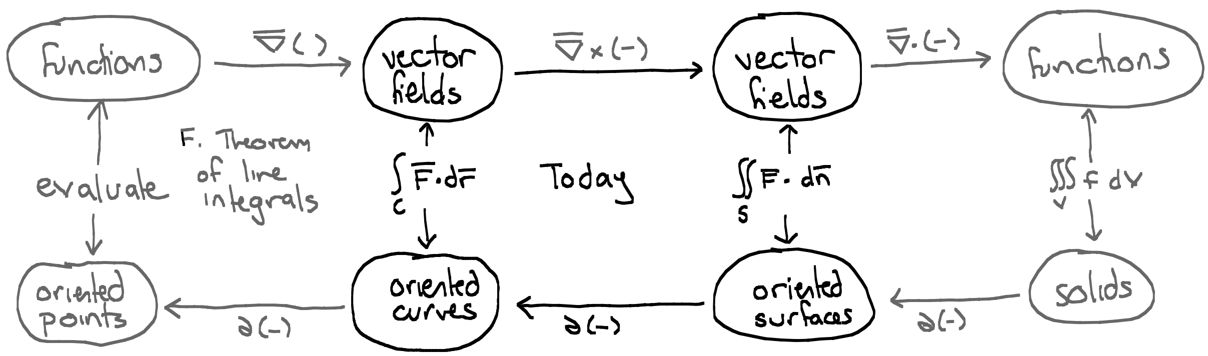
Lecture 22 – Green's Theorem

Note Title

Common Picture:



Today: Special Case of Stoke's Theorem: Green's Theorem

Def A simple closed curve is a closed curve that doesn't intersect itself.

In other words, if $\bar r(t)$ traces out the curve, $a \le t \le b$, then

$\bar r(a) = \bar r(b)$, and for $a < t_1 < t_2 < b$,

$\bar r(t_1) \ne \bar r(t_2)$.

A simple closed curve bounds a region, and we say the curve is positively oriented if the region is always to the left of the curve. $\longleftrightarrow$ The curve is oriented counter-clockwise.

Thm (Green's Theorem) Let C be a positively oriented, simple closed curve, enclosing a region D. Then

$$\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$\frac{\partial}{\partial x}$ before $\frac{\partial}{\partial y}$

We will often write $C = \partial D$, so this is essentially the middle box in our diagram.

Ex: C = square w/ vertices $(\pm 1, \pm 1)$, $\bar F = \langle \underbrace{e^{\tan x} - y}_{P}, \underbrace{x + e^{y^2}}_{Q} \rangle$

Then $\displaystyle\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_D 1 - {-1}\,dA = \boxed{8}$

Thus we can use double integrals to evaluate line integrals. Can also do reverse.

If C is the boundary of a region D, then

$$\text{Area}(D) = \int_C \tfrac{1}{2}(-y\,dx + x\,dy) = \int_C x\,dy = \int_C y\,dx$$

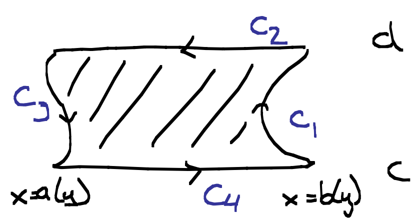
<u>Example</u>: Let C be the ellipse $x = 2\cos t$, $y = \sin t$, $0 \le t \le 2\pi$, and D the solid ellipse. Then

$$\text{Area } D = \int_C \tfrac{1}{2}(-y\,dx + x\,dy) = \int_0^{2\pi} \tfrac{1}{2}\left(-\sin t \cdot -2\sin t + 2\cos t \cdot \cos t\right) dt$$

$$= \int_0^{2\pi} 1\,dt = \boxed{2\pi}.$$

We'll prove Green's Theorem for a type II region. We will actually show that $\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$.

$$D = \left\{ \begin{array}{l} c \le y \le d \\ a(y) \le x \le b(y) \end{array} \right\}$$



$$\iint_D \frac{\partial Q}{\partial x}\,dA = \int_c^d \int_{a(y)}^{b(y)} \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d Q(x,y)\Big|_{x=a(y)}^{x=b(y)} dy$$

$$= \int_c^d Q(b(y),y)\,dy - \int_c^d Q(a(y),y)\,dy \qquad (*)$$

Now look at $\int_C Q\,dy$: Can break C into C_1, C_2, C_3, C_4:

$$\int_C Q\,dy = \int_{C_1} Q\,dy + \int_{C_2} Q\,dy + \int_{C_3} Q\,dy + \int_{C_4} Q\,dy$$

On C_2, C_4, y is constant $\Rightarrow dy = 0 \Rightarrow \int Q\,dy = 0$

On C_1: Can param. with respect to y:

$$\left.\begin{aligned} x &= b(y), \\ y &= y \end{aligned} \qquad c \le y \le d \right\} \Rightarrow \int_{C_1} Q\,dy = \int_c^d Q(b(y), y)\,dy$$

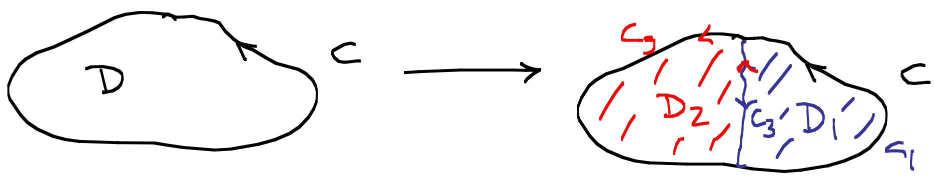
On C_3: C_3 is of the same form; we just run it backwards:

$$\int_{C_3} Q\,dy = -\int_c^d Q(a(y), y)\,dy$$

So $\displaystyle\int_C Q\,dy = \int_c^d Q(b(y), y)\,dy - \int_c^d Q(a(y), y)\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$ by (*)

Similar arguments work for type I regions. Why does this work?

① Can break up regions into type I or II regions:



Then $\left.\begin{aligned} \partial D_1 &= C_1 + C_3 \\ \partial D_2 &= C_2 + -C_3 \end{aligned}\right\} \quad \partial D = C_1 + C_2$

So
$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \int_C P\,dx + Q\,dy$$
$$\parallel \qquad\qquad\qquad \parallel$$
$$\iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \int_{C_1} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy$$
$$+ \qquad\qquad\qquad + \qquad\qquad \text{cancel}$$
$$\iint_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \int_{C_2} P\,dx + Q\,dy - \int_{C_3} P\,dx + Q\,dy$$

As a consequence, we can apply Green's theorem to regions with multiple boundary components.

Ex: $\displaystyle\int_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$ is the same for any path around $(0,0)$.

Let C_1 be a small circle around $(0,0)$. Then

$$0 = \iint_D \underbrace{\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)}_{0} dA = \int_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy - \int_{C_1} -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

Computing around C_1 is easy: $x = a\cos t$ $\quad y = a\sin t$

$$\Rightarrow \int_{C_1} -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \int_0^{2\pi} 1\,dt = 2\pi.$$