Lecture 19 - Spherical Coordinates & Vector Fields

Note Title

Ex: Convert $(\rho,\theta,\phi) = (2, \pi/4, \pi/3)$ to cylindrical & Cartesian:

$(\rho,\theta,\phi) \rightsquigarrow (r,\theta,z)$ $\qquad r = \rho\sin\phi$

$(\sqrt{3}, \pi/4, 1)$ $\qquad z = \rho\cos\phi$

$(r,\theta,z) \rightsquigarrow (x,y,z)$ $\qquad x = r\cos\theta$

$\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1\right)$ $\qquad y = r\sin\theta$

Naturally occurring example of spherical coords: latitude & longitude

$\rho =$ radius of earth (~ 3959 mi)

$\phi =$ "latitude" (measured from (x,y) plane)

$\theta =$ longitude $(-\pi \le \theta \le \pi)$.

This gives intuition: θ varies over 2π, but ϕ varies between 0 & π (90°N to 90°S).

$\Rightarrow$ Defaults for a region: $0 \le \rho$

$0 \le \theta < 2\pi$

$0 \le \rho \le \pi$

dV in Spherical:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} \rho & \theta & \phi \end{array} \\ \left| \begin{array}{ccc} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{array} \right| \end{array}$$

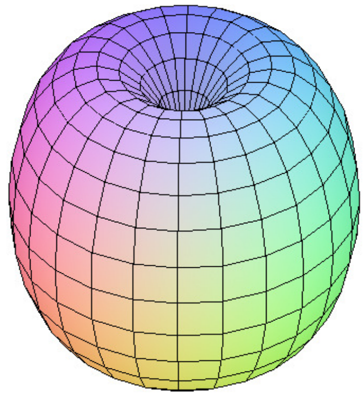
(Can factor out common terms: like $\bar{u} \times a\bar{v} = a(\bar{u}\times\bar{v})$)

$$= \rho^2\sin\phi\left(\underbrace{-\sin^2\phi\cos^2\theta \;\; - \cos^2\phi\cos^2\theta}_{-\cos^2\theta}\;\; \underbrace{- \cos^2\phi\sin^2\theta \;\; - \sin^2\phi\sin^2\theta}_{-\sin^2\theta}\right) = -\rho^2\sin\phi$$

So $dV = |\rho^2 \sin\phi| \, d\rho \, d\theta \, d\phi = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$ (w/ usual ϕ bounds)

<u>Ex</u>: $\rho = 2 \sin\phi$

(doughnut shape)



$\rho = 2\sin\phi \Rightarrow$

$\rho^2 = 2\rho \sin\phi \leftrightarrow$

$(r^2 + z^2) = 2r \leftarrow$ (cylindrical)

$\leftrightarrow (x^2+y^2+z^2)^2 = 2(x^2+y^2)$

Yuck!

The region is

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$0 \le \rho \le 2\sin\phi$

$\Rightarrow$ Volume given by $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$= 2\pi \int_0^{\pi} \frac{8\sin^4\phi}{3} \, d\phi = 2\pi^2$$

Can heuristically remember $\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$ as

$$dV = \rho \cdot z \cdot d\rho \, d\theta \, d\phi$$

$(\text{length})^3 = \text{length} \cdot \text{length} \cdot \text{length}$ nothing

All building up to study of vector fields & "Stoke's Theorem".

<u>Def</u>: A <u>vector field</u> is a function that assigns to each point of the plane a vector.

$$\vec{F}(x,y) = \langle f(x,y), g(x,y) \rangle = f(x,y)\vec{i} + g(x,y)\vec{j}$$

Very common in the natural world:

1. Wind: at each pt, wind has a velocity vector
   $\rightsquigarrow$ fun pictures of hurricanes & tornadoes as collections of arrows

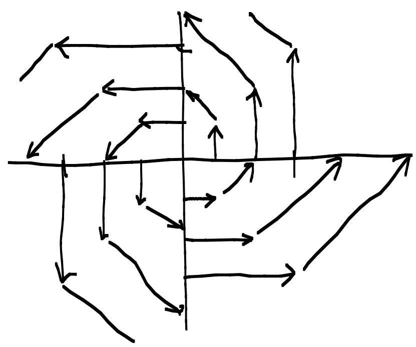
2. Fluid flow: again, each point has a velocity vector (near sides of a pipe, fluid is not moving, near center it flows fast)

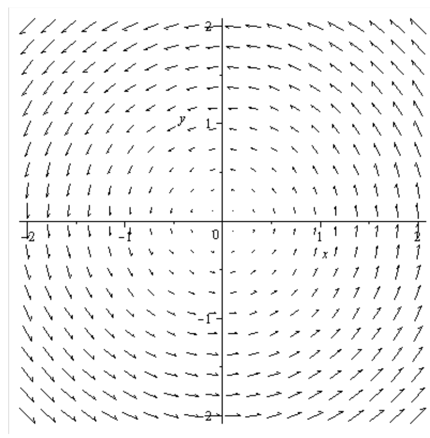
3. Schools of fish / people at a party form a vector field by looking at where they face.
4. Classical force fields (gravity, electric, etc.)

Easy & Useful to plot vector fields:

at a random sample of points, draw the vector $\bar{F}(x,y)$ w/ tail @ (x,y).

<u>Ex</u> $\bar{F} = \langle -y, x \rangle$





(this is scaled to give a better idea of more points)

Big example of vector fields: gradients:

If f is differentiable then $\bar{\nabla} f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle$

is a vector field. Such fields are called conservative

Last useful tool is "flow lines": the path a particle in the field moves $\longleftrightarrow$ Curves s.t. tangent at a point is the vector field.

<u>Ex</u>: $\bar{F} = \langle -y, x \rangle$ has flow lines circles oriented counterclockwise:

$$\left.\begin{array}{l} x = a\cos\theta \\ y = a\sin\theta \end{array}\right\} \Rightarrow \quad \bar{r}'(\theta) = \langle -a\sin\theta, a\cos\theta \rangle = \langle -y, x \rangle$$