Lecture 19 - Spherical Coordinates & Vector Fields

Ex: Convert \((\rho, \Theta, \Phi) = (2, \pi/4, \pi/3)\) to cylindrical & Cartesian:

\((\rho, \Theta, \Phi) \mapsto (r, \Theta, z)\)

\((2, \pi/4, 1)\)

\((r, \Theta, z) \mapsto (x, y, z)\)

(x, y, z)

\((\sqrt{3}, \pi/4, 1)\)

\(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1\)

Naturally occurring example of spherical coords: latitude & longitude

\(\rho = \) radius of earth (~ 39,59 mi)

\(\phi = \) "latitude" (measured from (xy) plane)

\(\Theta = \) longitude (-\(\pi \leq \Theta \leq \pi\)).

This gives intuition: \(\Theta\) varies over \(2\pi\), but \(\phi\) varies between 0 & \(\pi\) (90°N to 90°S).

\(\Rightarrow\) Defaults for a region:

\(0 \leq \rho\)

\(0 \leq \Theta < 2\pi\)

\(0 \leq \phi \leq \pi\)

\[dV = \rho^2 \sin \phi \, d\rho \, d\Theta \, d\phi\]

\(\frac{\partial (x,y,z)}{\partial (\rho,\Theta,\Phi)} = \begin{vmatrix} x & y & z \\ \sin \phi \cos \Theta & \sin \phi \sin \Theta & \cos \phi \\ -\rho \sin \phi \sin \Theta & \rho \sin \phi \cos \Theta & -\rho \sin \phi \end{vmatrix}\)

(Can factor out common terms: like \(\vec{u} \cdot \vec{v} = a(\vec{u} \cdot \vec{v})\))

\[= \rho^2 \sin \phi \left( \begin{array}{ccc} -\sin \phi \cos^2 \Theta & -\cos \phi \sin^2 \Theta & -\sin \phi \sin^2 \Theta & -\cos \phi \cos^2 \Theta \\ -\sin^2 \Theta & -\sin^2 \Theta & -\cos^2 \Theta \\ -\cos \phi \sin^2 \Theta & \cos \phi \cos^2 \Theta & -\sin \phi \sin^2 \Theta \\ -\cos \phi \cos^2 \Theta & -\sin \phi \cos^2 \Theta & \cos \phi \sin^2 \Theta \\ \end{array} \right) = -\rho^2 \sin \phi\]
So \[ dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \] (w/ usual \( \phi \) bounds)

**Ex:** \( \rho = 2 \sin \phi \)
(doughnut shape)

The region is:
0 \( \leq \theta \leq 2\pi \)
0 \( \leq \phi \leq \pi \)
0 \( \leq \rho \leq 2 \sin \phi \)

\[ \Rightarrow \text{Volume given by} \int_0^{2\pi} \int_0^\pi \int_0^{2\sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

\[ = 2\pi \int_0^\pi \frac{8 \sin \phi}{3} \, d\phi = 2\pi^2 \]

Can heuristically remember \( \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \) as:

\[ \int \int \int \rho \cdot 2 \cdot 3 \cdot \text{length} \cdot \text{length} \cdot \text{length} \]

\[ \text{nothing} \]

All building up to study of vector fields \& "Stoke's Theorem".

**Def:** A **vector field** is a function that assigns to each point of the plane a vector.

\[ \mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle = f(x, y)\mathbf{i} + g(x, y)\mathbf{j} \]

Very common in the natural world:

1. **Wind:** at each pt, wind has a velocity vector
   - Fun pictures of hurricanes \& tornadoes as collections of arrows
2. **Fluid flow:** again, each point has a velocity vector (near sides of a pipe, fluid is not moving, near center it flows fast)
3. Schools of fish / people at a party form a vector field by looking at where they face.

4. Classical force fields (gravity, electric, etc.)

Easy and useful to plot vector fields:

at a random sample of points, draw the vector $\mathbf{F}(x,y)$ w/ tail @ $(x,y)$.

**Ex** $\mathbf{F} = \langle -y, x \rangle$

Big example of vector fields: gradients:

If $f$ is differentiable, then $\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle$

is a vector field. Such fields are called **conservative**.

Last useful tool is "flow lines": the path a particle in the field moves $\leftrightarrow$ Curves s.t. tangent at a point is the vector field.

**Ex**: $\mathbf{F} = \langle -y, x \rangle$ has flow lines circles oriented counterclockwise.

$x = a \cos \theta$ \quad $y = a \sin \theta$ \quad $f'(\theta) = \langle -a \sin \theta, a \cos \theta \rangle = \langle -y, x \rangle$