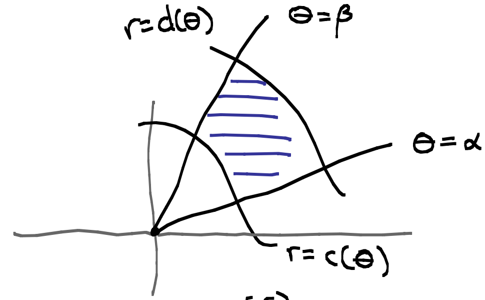
# Lecture 16 - Polar Integration & Change of Variables

Note Title

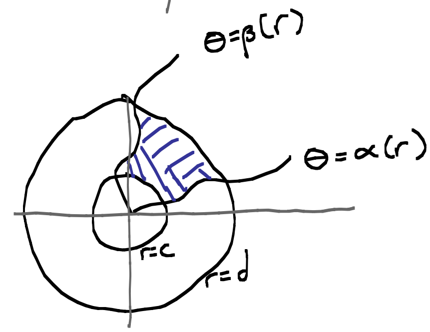
Saw last time: $dA = r\,dr\,d\theta$




Two types of regions:

Type I: $\alpha \le \theta \le \beta \qquad c(\theta) \le r \le d(\theta)$

$$\int_\alpha^\beta \int_{c(\theta)}^{d(\theta)} f(r,\theta)\, r\,dr\,d\theta$$




Type II: $c \le r \le d, \quad \alpha(r) \le \theta \le \beta(r)$

$$\int_c^d \int_{\alpha(r)}^{\beta(r)} f(r,\theta)\, r\,dr\,d\theta$$

Ex: $R = x^2+y^2 \le 9 \longleftrightarrow r \le 3, \quad 0 \le \theta < 2\pi$

$f(x,y) = \sqrt{9-x^2-y^2} \longleftrightarrow f(r,\theta) = \sqrt{9-r^2}$ } easier

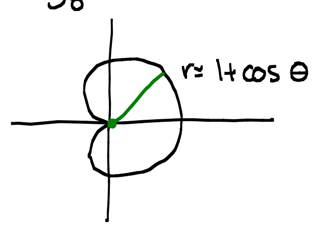
$$\iint_R F\,dA = \int_0^{2\pi}\int_0^3 \sqrt{9-r^2}\; r\,dr\,d\theta \quad \left(\text{vs} \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{9-x^2-y^2}\; dy\,dx\right)$$

Yuck!

$u = 9-r^2$
$du = -2r\,dr$

$$= \int_0^{2\pi}\int_9^0 -\tfrac{1}{2}\sqrt{u}\; du\,d\theta = \int_0^{2\pi} \tfrac{1}{3}(9)^{3/2}\,d\theta = \boxed{18\pi}$$

Ex: Area bound by $r = 1+\cos\theta$




Type I region:
$0 \le \theta < 2\pi$
$0 \le r \le 1+\cos\theta$

$$\text{Area} = \iint_R 1\,dA$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{2\pi} \tfrac{1}{2}r^2\Big|_0^{1+\cos\theta} d\theta = \int_0^{2\pi} \tfrac{1}{2}(1+\cos\theta)^2\,d\theta$$

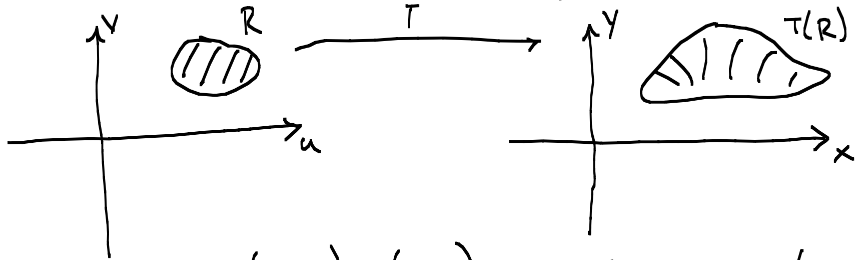
$$= \tfrac{1}{2}\int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta = \tfrac{1}{2}\int_0^{2\pi} \tfrac{3}{2} + 2\cos\theta - \cos 2\theta\; d\theta = \boxed{\tfrac{3\pi}{2}}$$

This technique can be used to recover the Calc II area formulas.

# Change of Variables

We want a way to find $\iint_R f\,dA$ if we pick new coords.

What are new coords? A pair of functions of 2 variables:



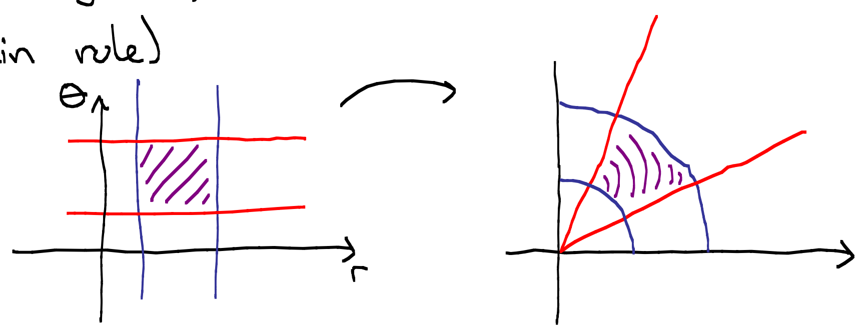

$$T(u,v) = (x,y) \longleftrightarrow \left.\begin{array}{l} x = x(u,v) \\ y = y(u,v) \end{array}\right\} \text{ differentiable}$$

$$T(u,v) = \langle x(u,v), y(u,v) \rangle$$

(same idea from chain rule)

Motivating idea: polar

$x = r\cos\theta$
$y = r\sin\theta$

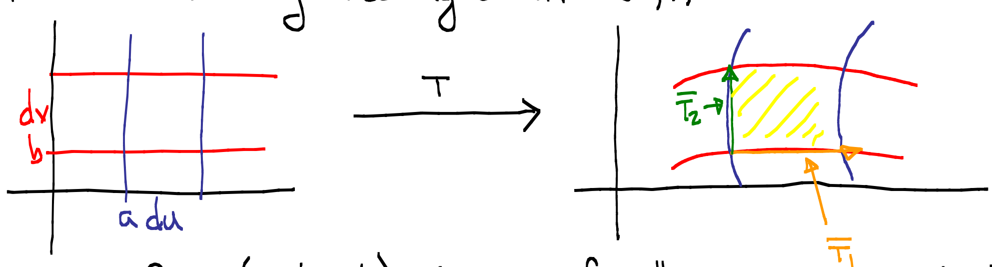



Have to do 2 things:

1. Understand $dA$
2. Figure out the $(u,v)$-plane region.

Look at 1.

Start with a tiny rectangle in $(u,v)$:




The area of T(rectangle) $\approx$ area of llogram spanned by $\overline{T_1}$ & $\overline{T_2}$

$\overline{T_1}$ = differential tangent to "v=b" curve: $= \frac{d}{du}\left\langle x(u,b), y(u,b)\right\rangle \cdot du$

$= \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle du$

$\overline{T_2}$ = differential tangent to "u=a" curve: $= \frac{d}{dv}\left\langle x(a,v), y(a,v)\right\rangle \cdot dv$

$= \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle dv$

$$\text{Area} = \left| \bar{T}_1 \times \bar{T}_2 \right| = \left\| \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du & 0 \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv & 0 \end{vmatrix} \right\| = \left| \left( \frac{\partial x}{\partial u}du \frac{\partial y}{\partial v}dv - \frac{\partial x}{\partial v}dv \frac{\partial y}{\partial u}du \right) \bar{k} \right|$$

$$= \left| \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u} \right| du\,dv$$

__Def__ The Jacobian of $T$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{array}{c|cc|} & u & v \\ x & \partial x/\partial u & \partial x/\partial v \\ y & \partial y/\partial u & \partial y/\partial v \end{array}$$

The rows are labeled by $x, y$ the cols by the new variables

__Remark:__ This is the determinant of a matrix called __the__ derivative of $T$. This captures all of the "linear" information of $T$.

__Thm__

$$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

This is because $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right|$

__Ex__ $x = r\cos\theta$
$y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{array}{c|cc|} & r & \theta \\ x & \cos\theta & -r\sin\theta \\ y & \sin\theta & r\cos\theta \end{array} = r\cos^2\theta + r\sin^2\theta = r$$

$\Rightarrow$ $dA = r\,dr\,d\theta$!