Lecture 12 – Maxima & Minima

Note Title

Look for places where the tangent plane is horizontal.

$\leftrightarrow$ equation of the plane is of the form $z - c = 0$.

The equation of the tangent plane is

$$z - f(a,b) = \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b)$$

$\Rightarrow$ tangent plane is horizontal iff

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0.$$

__Def__ (a,b) is a __critical point__ if $\vec{\nabla} f(a,b) = \vec{0}$.

__Ex__: $f(x,y) = x^3 - 3x + y^3 - 3y$

$$\frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3 = 0 \Rightarrow y = \pm 1$$

Critical points:

$(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$.

Since the gradient points in the direction of steepest ascent, at critical points, there is no such direction.

__Def__ A function f has a __local max__ at (a,b) if

$$f(a,b) \geq f(c,d)$$

for all (c,d) close to (a,b).

A function f has a __local min__ at (a,b) if

$$f(a,b) \leq f(c,d)$$

for all (c,d) close to (a,b).

Maxima & minima occur at critical points.

__Ex__ $f(x,y) = xy$: $\frac{\partial f}{\partial x} = y = 0 \Rightarrow y = 0$ $\quad \frac{\partial f}{\partial y} = x = 0 \Rightarrow x = 0$

There is one critical point: $(0,0)$

This point is not a maximum or minimum. It looks like a saddle.

In the 1-var case, have the second derivative test:

$$\text{If} \quad f'(a)=0 \Rightarrow \begin{cases} f''(a) > 0 \Rightarrow \min \\ f''(a) < 0 \Rightarrow \max \end{cases}$$

In the 2-var case, have more directions to worry about:

$$\frac{\partial^2 f}{\partial x^2}(a,b) \begin{cases} > 0 & \text{curve is concave up in } x\text{-direction} \\ < 0 & \text{curve is concave down in } x\text{-direction} \end{cases}$$

$$\frac{\partial^2 f}{\partial y^2}(a,b) \begin{cases} > 0 & \text{curve is concave up in } y\text{-direction} \\ < 0 & \text{curve is concave down in } y\text{-direction} \end{cases}$$

Let $D = \left(\frac{\partial^2 f}{\partial x^2}\right) \cdot \left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

<u>Thm</u> If (a,b) is a critical point then at (a,b), f has a

<u>maximum</u> $\qquad D > 0, \ \frac{\partial^2 f}{\partial x^2}(a,b) < 0$

<u>minimum</u> $\quad$ if $\quad D > 0, \ \frac{\partial^2 f}{\partial x^2}(a,b) > 0$

<u>saddle point</u> $\qquad D < 0$

$-\left(\frac{\partial^2 f}{\partial x \partial y}\right)$ is <u>always</u> less than or equal to zero,

if $D > 0$, then $\left(\frac{\partial^2 f}{\partial x^2}\right) \cdot \left(\frac{\partial^2 f}{\partial y^2}\right) > 0$.

Thus if $D > 0$, $\frac{\partial^2 f}{\partial x^2}$ & $\frac{\partial^2 f}{\partial y^2}$ have the same sign.

$\Rightarrow$ x-curves & y-curves have same concavity.

Can check sign of <u>either</u> $\frac{\partial^2 f}{\partial x^2}$ or $\frac{\partial^2 f}{\partial y^2}$.

<u>Ex</u>: $f(x,y) = xy$. Saw critical point at $(0,0)$.

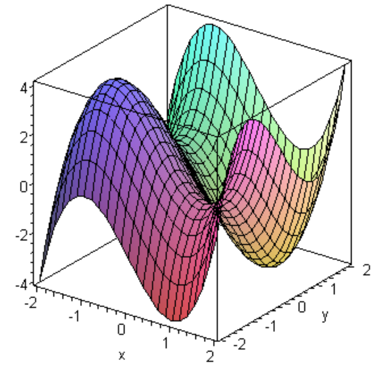
$$\frac{\partial^2 f}{\partial x^2} = 0 = \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = 1, \quad \text{so} \quad D = -1.$$

This is a saddle.

Ex: $f(x,y) = x^3 - 3x + y^3 - 3y$ c.p.: $(\pm 1, \pm 1)$

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial x \partial y} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 6y \quad \Rightarrow \quad D = 36xy$$

c.p.	D	$\partial^2 f/\partial x^2$	max? min? saddle?
$(1,1)$	$36 > 0$	$6 > 0$	min
$(1,-1)$	$-36 < 0$	—	saddle
$(-1,1)$	$-36 < 0$	—	saddle
$(-1,-1)$	$36 > 0$	$-6 < 0$	max

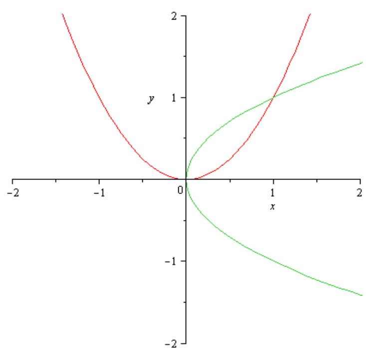


Ex: $f(x,y) = x^3 - 3xy + y^3$

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x} = 3x^2 - 3y = 0 \\ \dfrac{\partial f}{\partial y} = 3y^2 - 3x = 0 \end{array}\right\}$$ must be true simultaneously:

$x^2 = y$

$y^2 = x$



$x^2 = y \Rightarrow x = y^2 = (x^2)^2$

$x = x^4 \Rightarrow x^4 - x = 0$

$x(x-1)(x^2+x+1) = 0 \Rightarrow$

$x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$

2 c.p.: $(0,0)$; $(1,1)$

$$\left.\begin{array}{l} \dfrac{\partial^2 f}{\partial x^2} = 6x \\ \dfrac{\partial^2 f}{\partial x \partial y} = -3 \\ \dfrac{\partial^2 f}{\partial y^2} = 6y \end{array}\right\} \Rightarrow D = (6x)(6y) - (-3)^2 = 36xy - 9$$

c.p.	D	$\partial^2 f/\partial x^2$	?
$(0,0)$	$-9 < 0$	—	saddle
$(1,1)$	$27 > 0$	$6 > 0$	min