Lecture 10 - Chain Rule

Note Title

Saw last time how to describe the tangent plane. Now we'll focus on seeing what tangents to other curves look like.

<u>Ex</u>: $f(x,y) = x^2 + y^3$

Look at tangent vectors to curves $x = t^n$, $y = t^m$, $z = f(x,y)$

$(n,m) = (1,2)$: $\bar{r}(t) = \langle t, t^2, t^2 + t^6 \rangle \Rightarrow \bar{r}'(t) = \langle 1, 2t, 2t + 6t^5 \rangle$

$(n,m) = (3,1)$: $\bar{r}(t) = \langle t^3, t, t^6 + t^3 \rangle \Rightarrow \bar{r}'(t) = \langle 3t^2, 1, 6t^5 + 3t^2 \rangle$

$(n,m) = (n,m)$: $\bar{r}(t) = \langle t^n, t^m, t^{2n} + t^{3m} \rangle \Rightarrow \bar{r}'(t) = \langle nt^{n-1}, mt^{m-1}, 2nt^{2n-1} + 3mt^{3m-1} \rangle$

In each case, the third coord of $\bar{r}'(t)$ can be recovered from

$x(t), y(t), x'(t), y'(t), \frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$:

$$2t + 6t^5 = 1 \cdot 2(t) + (2t) \cdot 3(t^2)^2$$

$$\vdots \qquad \vdots$$

$$2nt^{2n-1} + 3mt^{3m-1} = (nt^{n-1}) \cdot 2(t^n) + (mt^{m-1}) \cdot 3(t^m)^2$$

$$\frac{df}{dt} = \left(x'(t) \cdot \frac{\partial f}{\partial x}(x(t),y(t))\right) + \left((y'(t)) \cdot \frac{\partial f}{\partial y}(x(t),y(t))\right)$$

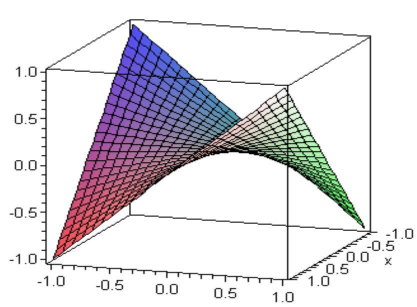
This is the chain rule:

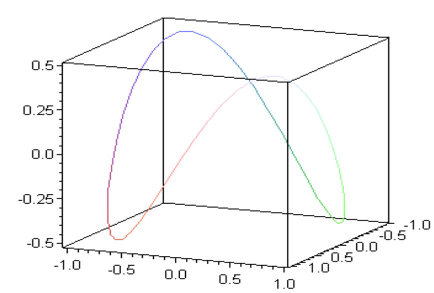
If $x = x(t)$, $y = y(t)$, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}(x(t),y(t)) \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t),y(t)) \cdot \frac{dy}{dt}$$

<u>Ex</u>: $f(x,y) = xy \qquad x = \cos t \qquad y = \sin t$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = y \cdot (-\sin t) + x \cdot (\cos t) = \cos^2 t - \sin^2 t$$





At $t = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4,$ this is zero $\Rightarrow$ peaks and valleys.

Aside: this is another way to understand surfaces: pick a family of curves filling the plane & look at the piece of the surface above each.

The chain rule works for any number of variables:

f a function of x, y, z

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

etc.

Why $\frac{df}{dt}$ rather than $\frac{\partial f}{\partial t}$? There is just <u>one</u> independent variable: t. So it is exactly like calc I: ordinary derivatives.

We can prove the chain rule with differentials:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Since $dx = \frac{dx}{dt}dt$ & $dy = \frac{dy}{dt}dt$,

$$df = \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right)dt \Rightarrow \text{chain rule.}$$

What happens if x & y are themselves functions of 2 variables?

Nothing changes: partial derivatives don't see other variables.

2 variable Chain Rule:

If $x = x(s,t)$, $y = y(s,t)$, then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

<u>Ex</u>: $f(x,y) = xy \qquad x = t^2 + s^2 \qquad y = t^2 - s^2$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = y\cdot 2s + x\cdot(-2s) = 2st^2 - 2s^3 + (-2st^2 - 2s^3) = -4s^3$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = y\cdot 2t + x\cdot 2t = 2t^3 - 2ts^2 + 2t^3 + 2ts^2 = 4t^3$$

We can always check by plugging in and computing directly.

$$f(x,y) = (t^2+s^2)(t^2-s^2) = t^4 - s^4.$$

<u>Ex</u>: $f(x,y) = x^2+y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = 2x\cdot\cos\theta + 2y\sin\theta = 2r\cos^2\theta + 2r\sin^2\theta = 2r$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = 2x\cdot(-r\sin\theta) + 2y(r\cos\theta) = -2r^2\cos\theta\sin\theta + 2r^2\cos\theta\sin\theta = 0$$

Again $f = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

so these make sense.

Get another way to look at implicit differentiation:

$f(x,y,z) = 0$ is a surface, defining z implicitly.

Then x, y, and z are all functions of ... x & y.

Apply $\frac{\partial}{\partial x}$:

$$\left.\begin{aligned} &\frac{\partial f}{\partial x} = 0 \\ &\| \\ &\frac{\partial f}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial x}}_{=1} + \cancel{\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial x}}^{\,0} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}\end{aligned}\right\} \Rightarrow \frac{\partial z}{\partial x} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

y doesn't see x

Apply $\frac{\partial}{\partial y}$:

$$\left.\begin{aligned} &\frac{\partial f}{\partial y} = 0 \\ &\| \\ &\cancel{\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial y}}^{\,0} + \frac{\partial f}{\partial y}\cdot\underbrace{\frac{\partial y}{\partial y}}_{=1} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}\end{aligned}\right\} \Rightarrow \frac{\partial z}{\partial y} = \frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

x doesn't see y