MATH 231, Calculus III, Section 1, Fall 2007

Final Exam

Date: Dec. 11, Tuesday
Time: 9:00 am – 12:00 noon

Please write clearly, reduce answers to their simplest form, and box your answers.
To receive full credit you must show ALL your work.

Student's Name (Please print): ____

Pledge: On my honor as a student at the University of Virginia I have neither given nor received aid on this test.

Signature: ______________________________

Problem	**Points**	**Score**
1	15	
2	15	
3	20	
4	20	
5	25	
6	25	
7	20	
8	20	
9	10	
10	15	
11	15	
12	25	
13	25	
14	25	
15	25	
16 (Bonus)	25	
Total	**325/300**	

Problem 1 (15 points)
(a) (5 points) For what value(s) of t are $\mathbf{a} = \langle t+2, t, t\rangle$ and $\mathbf{b} = \langle t-2, t+1, 1\rangle$ orthogonal?

(b) (10 points) Consider the surface S given by $\rho^2 - 2\rho\cos\phi = 3$ in spherical coordinates. Convert the equation into an equation in rectangular coordinates and identify (describe) the surface.

Problem 2 (15 points)
Show that $\mathbf{F}$ is a conservative vector field, where
(a) (5 points) $\mathbf{F}(x, y) = (2 + 3x^2y)\mathbf{i} + (x^3 - 3y^2)\mathbf{j}$.

(b) (10 points) $\mathbf{F}(x, y, z) = (e^y + z^2)\mathbf{i} + xe^y\mathbf{j} + 2xz\mathbf{k}$.

Problem 3 (20 points)
(a) (15 points) Sketch the surface S that is part of $x^2 + y^2 + z^2 = 4$ which lies above $z = 1$. Give the parametrization of S in terms of spherical angles ϕ and θ. Indicate the parameter domain of this parametrization.

(b) (5 points) Give the parametrization of S in terms of x and y. Indicate the parameter domain of this parametrization.

Problem 4 (20 points)
Find the directional derivative of $f(x, y) = \ln(x^2 + y^2)$ at the point $(2, 1)$ in the direction of $\mathbf{v} = \langle -1, 2 \rangle$.

Problem 5 (25 points)
Sketch the region D in $\mathbb{R}^2$ bounded by $y = 2x^2 - 10$ and $y = -2x + 2$, then set up the integral

$$I = \iint\limits_D f(x, y)\, dA$$

in two different ways.

Problem 6 (25 points)
Evaluate the integral

$$I = \iint\limits_{R} \frac{x+y}{x-y}\, dA$$

by making an appropriate change of variables, where R is the rectangle enclosed by $x+y=0$, $x+y=2$, $x-y=-3$ and $x-y=-1$.

Problem 7 (20 points)
Express the vector $\mathbf{b} = 2\mathbf{i} + 8\mathbf{j} - \mathbf{k}$ as the sum of two vectors, one parallel to $\mathbf{a} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and the other perpendicular to $\mathbf{a}$.

Problem 8 (20 points)
Find the maximum and minimum values of the function $f(x, y, z) = 8x - 4z$ subject to the constraint $x^2 + 10y^2 + z^2 = 5$.

Problem 9 (10 points)
Find the velocity, acceleration and speed of the particle whose position function is given by $\mathbf{r}(t) = e^{2t}\,\mathbf{i} + \sqrt{t}\,\mathbf{j}$.

Problem 10 (15 points)
Find the limit, if it exists, or show that the limit does not exist:

$$L = \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}.$$

Problem 11 (15 points)
Find the (most general) potential function for $\mathbf{F}(x,y) = (-3\sin x + 2xy^2)\,\mathbf{i} + (x^2+1)2y\,\mathbf{j}$.

Problem 12 (25 points)
Sketch the region E given by $x^2 + y^2 \leq z \leq 4$, then use the Divergence Theorem to evaluate

$$I = \iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z(x^2 + y^2)\mathbf{k}$ and S is the boundary of the region E.

Problem 13 (25 points)
Sketch the surface S that is part of $x^2 + y^2 = 4$ which lies between $z = 0$ and $z = 3$. Using a parametrization of S (specify a parameter domain), find an equation of the tangent plane to S at the point $(\sqrt{3}, 1, 2)$.

Problem 14 (25 points)
Use Stokes' Theorem to evaluate

$$I = \iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = -yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and S is part of $x^2 + y^2 + z^2 = 9$ that lies inside $x^2 + y^2 = 4$ and above the xy-plane, with normal vector pointing in the positive z-direction.

Problem 15 (25 points)
Find the surface area of S, where S is a torus (doughnut) given by

$$\mathbf{r}(u, v) = \langle (2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u \rangle$$

with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$.

Bonus Problem 16 (25 points)
Show that our definition of the surface area is consistent with the surface area formula from single-variable calculus.