The Frenet-Serret Formulas

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We start with the formula we know by the definition:

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}.$$

We also defined

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

We know that $\mathbf{B}$ is a unit vector, since $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors. This means, just as for $\mathbf{T}$, that

$$\mathbf{B} \cdot \frac{d\mathbf{B}}{ds} = 0.$$

Now we differentiate:

$$\frac{d}{ds}\mathbf{B} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = \left(\frac{d\mathbf{T}}{ds}\right) \times \mathbf{N} + \mathbf{T} \times \left(\frac{d\mathbf{N}}{dx}\right).$$

Since the derivative on $\mathbf{T}$ is in the same direction as $\mathbf{N}$, we know that the first cross product is $\mathbf{0}$. The second one we can't say much about. We do, however, know that it is orthogonal to $\mathbf{T}$, since $\mathbf{T}$ is one of the factors. In other words, we know that

$$\frac{d}{ds}\mathbf{B} \cdot \mathbf{B} = 0 = \frac{d}{ds}\mathbf{B} \cdot \mathbf{T}.$$

This means that it has to be in the direction of $\mathbf{N}$, and we just define the magnitude to be $-\tau$. In other words, we see that

$$\frac{d}{ds}\mathbf{B} = -\tau\mathbf{N}.$$

Now we approach the derivative of $\mathbf{N}$. We can write $\mathbf{N} = \mathbf{B} \times \mathbf{T}$, just by thinking about the right hand rule. Now we differentiate this, using the previous rules:

$$\frac{d\mathbf{N}}{ds} = \left(\frac{d\mathbf{B}}{ds}\right) \times \mathbf{T} + \mathbf{B} \times \left(\frac{d\mathbf{T}}{ds}\right) = -\tau\mathbf{N} \times \mathbf{T} + \mathbf{B} \times \kappa\mathbf{N}.$$

Now we just rewrite the cross products using the right hand rule:

$$-\tau\mathbf{N} \times \mathbf{T} + \mathbf{B} \times \kappa\mathbf{N} = \tau\mathbf{B} - \kappa\mathbf{T}.$$