The Frenet-Serret Formulas

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We start with the formula we know by the definition:

\[ \frac{dT}{ds} = \kappa N. \]

We also defined

\[ B = T \times N. \]

We know that \( B \) is a unit vector, since \( T \) and \( N \) are orthogonal unit vectors. This means, just as for \( T \), that

\[ B \cdot \frac{dB}{ds} = 0. \]

Now we differentiate:

\[ \frac{d}{ds} B = \frac{d}{ds} (T \times N) = \left( \frac{dT}{ds} \right) \times N + T \times \left( \frac{dN}{dx} \right). \]

Since the derivative on \( T \) is in the same direction as \( N \), we know that the first cross product is \( 0 \). The second one we can’t say much about. We do, however, know that it is orthogonal to \( T \), since \( T \) is one of the factors. In other words, we know that

\[ \frac{d}{ds} B \cdot B = 0 = \frac{d}{ds} B \cdot T. \]

This means that it has to be in the direction of \( N \), and we just define the magnitude to be \( -\tau \). In other words, we see that

\[ \frac{d}{ds} B = -\tau N. \]

Now we approach the derivative of \( N \). We can write \( N = B \times T \), just by thinking about the right hand rule. Now we differentiate this, using the previous rules:

\[ \frac{dN}{ds} = \left( \frac{dB}{ds} \right) \times T + B \times \left( \frac{dT}{ds} \right) = -\tau N \times T + B \times \kappa N. \]

Now we just rewrite the cross products using the right hand rule:

\[ -\tau N \times T + B \times \kappa N = \tau B - \kappa T. \]