NAME: Solutions INSTRUCTOR: ______

Instructions: Write clearly. **You must show all work to receive credit.**

Missed: pg1 ___ pg2 ___ pg3 ___ pg4 ___ pg5 ___ pg6 ___ pg7 ___

pg8 ___ pg9 ___ pg10 ___ pg11 ___ pg12 ___ Total Score ___/300

1. (10 points) Set up the partial fraction decomposition of the following rational function. Do not solve for the coefficients.

$$\frac{5x^2 - 6x + 2008}{(x-1)(5x+3)^2(x^2+4)(x^2+x+1)^2}$$

← highest deg. in denominator (no long division nec.)

$$\boxed{\frac{A}{x-1} + \frac{B}{5x+3} + \frac{C}{(5x+3)^2} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{x^2+x+1} + \frac{Hx+I}{(x^2+x+1)^2}}$$

2. (10 points each) Evaluate the following definite integrals.

(a) $\int_0^{\pi/2} x^2 \sin(x)\, dx$

let $u = x^2$ & $dv = \sin(x)\,dx$
$\Rightarrow du = 2x$ $\quad v = -\cos(x)$

$= -x^2\cos(x)\Big|_0^{\pi/2} + 2\int_0^{\pi/2} x\cos(x)\, dx$

$= -x^2\cos(x)\Big|_0^{\pi/2} + 2\left[x\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin(x)\,dx\right]$

let $u = x$ & $dv = \cos(x)\,dx$
$\Rightarrow du = dx$ $\quad v = \sin(x)$

$= -x^2\cos(x) + 2x\sin(x) + 2\cos(x)\Big|_0^{\pi/2}$

$= -\frac{\pi^2}{4}\underbrace{\cos(\pi/2)}_{=0} + 2\left(\frac{\pi}{2}\right)\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} + 2\underbrace{\cos(\pi/2)}_{=0} + 0 - 0 - 2\underbrace{\cos(0)}_{=1}$

$= \boxed{\pi - 2}$

(b) $\displaystyle\int_0^3 \frac{5x}{(x^2-1)^{2/3}}\,dx$ ← Improper!

$\displaystyle\int \frac{5x}{(x^2-1)^{2/3}}\,dx$ let $u=x^2-1$, $du = 2x\,dx$

$= \frac{5}{2}\int u^{-2/3}\,du$

$= \frac{5}{2}\cdot 3\,u^{1/3}$

$= \frac{15}{2}(x^2-1)^{1/3}+C$

$$= \lim_{t\to 1^-}\int_0^t \frac{5x}{(x^2-1)^{2/3}}\,dx + \lim_{t\to 1^+}\int_t^3 \frac{5x}{(x^2-1)^{2/3}}\,dx$$

$$= \lim_{t\to 1^-} \frac{15}{2}(x^2-1)^{1/3}\Big|_0^t + \lim_{t\to 1^+} \frac{15}{2}(x^2-1)^{1/3}\Big|_t^3$$

$$= \lim_{t\to 1^-} \frac{15}{2}\Big[\underbrace{(t^2-1)^{1/3}}_{\to 0} - (-1)^{1/3}\Big] + \lim_{t\to 1^+} \frac{15}{2}\Big[8^{1/3} - \underbrace{(t^2-1)^{1/3}}_{\to 0}\Big]$$

$$= \frac{15}{2} + \frac{15}{2}(2)$$

$$= \frac{15}{2} + 15$$

$$= \boxed{\frac{45}{2}}$$

3. (10 points each) Compute the following indefinite integrals.

(a) $\displaystyle\int \frac{1}{x^2+2x+2}\,dx$

$x^2+2x+2 = x^2+2x+1+1$
$= (x+1)^2+1$

$$= \int \frac{1}{(x+1)^2+1}\,dx$$

let $u = x+1$
$\Rightarrow du = dx$

$$= \int \frac{1}{u^2+1}\,du$$

$$= \arctan(u)$$

$$= \boxed{\arctan(x+1)+C}$$

(b) $\int \frac{x^2}{(1-x^2)^{3/2}}\,dx$

let $x = \sin\theta$

$\Rightarrow dx = \cos\theta \, d\theta$

& $1-x^2 = 1-\sin^2\theta = \cos^2\theta$

$= \int \frac{\sin^2\theta}{(\cos^2\theta)^{3/2}} \cdot \cos\theta \, d\theta$

$= \int \frac{\sin^2\theta}{\cos^3\theta} \cos\theta \, d\theta$

$= \int \frac{\sin^2\theta}{\cos^2\theta} \, d\theta$

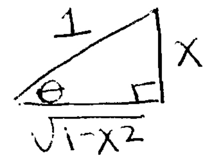
$= \int \tan^2\theta \, d\theta$

$= \int \sec^2\theta - 1 \, d\theta$

$= \tan\theta - \theta + C$

$= \boxed{\frac{x}{\sqrt{1-x^2}} - \arcsin(x) + C}$

if $x = \sin\theta = \frac{opp}{hyp}$



$\Rightarrow \tan\theta = \frac{x}{\sqrt{1-x^2}}$

& $\theta = \arcsin(x)$

4. (15 points) Compute the arc length of the curve $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ on the interval $1 \le x \le 5$.

$L = \int_1^5 \sqrt{1+[f'(x)]^2}\, dx$

$= \int_1^5 \sqrt{\frac{(x^2+1)^2}{4x^2}}\, dx$

$= \int_1^5 \frac{x^2+1}{2x}\, dx$

$= \int_1^5 \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{x}\, dx$

$= \frac{1}{4}x^2 + \frac{1}{2}\ln|x| \Big|_1^5$

$= \frac{1}{4}(25) + \frac{1}{2}\ln(5) - \frac{1}{4} - \frac{1}{2}\ln(1)$

$= \boxed{6 + \frac{1}{2}\ln(5)}$

$f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$

$\Rightarrow f'(x) = \frac{1}{2}x - \frac{1}{2x}$

$= \frac{1}{2}\left(x - \frac{1}{x}\right)$

$= \frac{1}{2}\left(\frac{x^2-1}{x}\right)$

$\Rightarrow 1+[f'(x)]^2 = 1 + \frac{(x^2-1)^2}{4x^2}$

$= \frac{4x^2 + x^4 - 2x^2 + 1}{4x^2}$

$= \frac{x^4+2x^2+1}{4x^2}$

$= \frac{(x^2+1)^2}{4x^2}$

5. (5 points each) Complete the following **definitions**:

   (a) The improper integral $\int_a^\infty f(x)\,dx$ is **convergent** if

   $\lim_{t\to\infty} \int_a^t f(x)\,dx$ exists

   (b) The improper integral $\int_a^\infty f(x)\,dx$ is **divergent** if

   $\lim_{t\to\infty} \int_a^t f(x)\,dx$ does not exist.

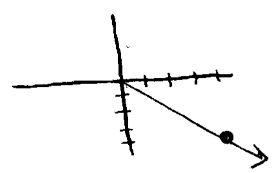
6. (10 points) Set up, but do not evaluate, an integral to compute the surface area of the solid of revolution generated by revolving the curve $f(x) = \cos(x)$, $0 \le x \le \pi/2$, about the y-axis.

$$A = \int_0^{\pi/2} 2\pi x \sqrt{1+[f'(x)]^2}\,dx$$
$$= \boxed{\int_0^{\pi/2} 2\pi x \sqrt{1+\sin^2 x}\,dx}$$

7. (5 points each) Find polar coordinates (r, θ) for the point with Cartesian coordinates $(x, y) = (4, -4)$ such that

   (a) $r > 0$:

   $r^2 = x^2+y^2 = 16+16 = 32$
   $\theta = \arctan(\frac{y}{x}) = \arctan(-1) = -\pi/4$



   $(r,\theta) = (\underline{\sqrt{32}, -\pi/4})$

   (b) $r < 0$:

   $(r,\theta) = (\underline{-\sqrt{32}, 3\pi/4})$

8. Consider the parametric curve defined by the equations $x(t) = \cos^3(t)$, $y(t) = \sin^3(t)$, $0 \le t \le \pi/2$.

(a) (15 points) Write the equation of the tangent line to the curve at the point where $t = \pi/4$.

$$m_{tan} = \frac{dy/dt}{dx/dt} = \frac{3\sin^2(t)\cos(t)}{-3\cos^2(t)\sin(t)} = -\frac{\sin(t)}{\cos(t)}$$

$$\Rightarrow m_{tan}\Big|_{t=\pi/4} = -\frac{\sin(\pi/4)}{\cos(\pi/4)} = -1$$

when $t = \pi/4$, $x = \cos^3(\pi/4) = \left(\frac{\sqrt{2}}{2}\right)^3 = \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{4}$

$y = \sin^3(\pi/4) = \frac{\sqrt{2}}{4}$

using $y - y_1 = m(x - x_1)$

$$\boxed{y - \frac{\sqrt{2}}{4} = -\left(x - \frac{\sqrt{2}}{4}\right)}$$

(b) (15 points) Compute the length of the parametric curve.

$$L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^2(t)\sin^2(t)}\, dt$$

$$= 3\int_0^{\pi/2} \cos(t)\sin(t)\, dt$$

$$= \frac{3}{2}\sin^2(t)\Big|_0^{\pi/2}$$

$$= \frac{3}{2}\left[\sin^2(\pi/2) - \sin^2(0)\right]$$

$$= \boxed{\frac{3}{2}}$$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$

$= 9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)$

$= 9\cos^2(t)\sin^2(t)\left[\cos^2(t) + \sin^2(t)\right]$

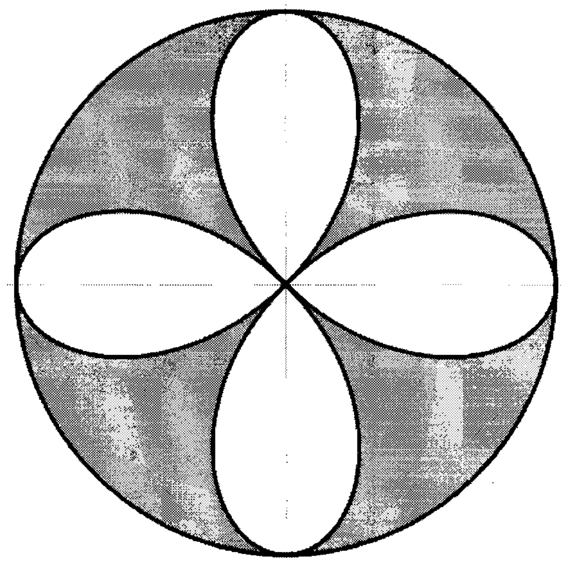
$= 9\cos^2(t)\sin^2(t)$

let $u = \sin(t)$

$\Rightarrow du = \cos(t)\, dt$

$\Rightarrow \int u\, du = \frac{1}{2}u^2$

9. (15 points) Find the area of the shaded region below, inside the polar curve $r = 2$ and outside the polar curve $r = 2\cos(2\theta)$.



$$A = \pi \cdot 2^2 - \int_0^{2\pi} \tfrac{1}{2}(2\cos(2\theta))^2\, d\theta$$

$$= 4\pi - 2\int_0^{2\pi} \cos^2(2\theta)\, d\theta$$

$$= 4\pi - 2\int_0^{2\pi} \tfrac{1}{2}(1+\cos(4\theta))\, d\theta$$

$$= 4\pi - \int_0^{2\pi} 1+\cos(4\theta)\, d\theta$$

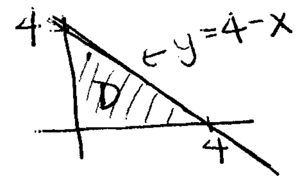
$$= 4\pi - \theta - \tfrac{1}{4}\sin(4\theta)\Big|_0^{2\pi}$$

$$= 4\pi - 2\pi - \tfrac{1}{4}\sin(8\pi) + 0 + 0$$

$$= \boxed{2\pi}$$

10. (15 points each) Evaluate the following double integrals.

(a) $\iint_D xe^y \, dA$ where D is the region bounded by the curves $y = 4 - x$, $y = 0$, and $x = 0$.



$= \int_0^4 \int_0^{4-x} xe^y \, dy \, dx$

$= \int_0^4 xe^y \Big|_{y=0}^{y=4-x} dx$

$= \int_0^4 xe^{4-x} - x \, dx$

let $u = x$ & $dv = e^{4-x} dx$
$\Rightarrow du = dx$ $v = -e^{4-x}$

$= \int_0^4 xe^{4-x} dx - \int_0^4 x \, dx$

$= -xe^{4-x} \Big|_0^4 + \int_0^4 e^{4-x} dx - \frac{1}{2}x^2 \Big|_0^4$

$= -xe^{4-x} - e^{4-x} - \frac{1}{2}x^2 \Big|_0^4$

$= -4 - 1 - \frac{16}{2} + 0 + e^4 + 0$

$= \boxed{e^4 - 13}$

(b) $\int_0^1 \int_0^{\ln(3)} xye^{xy^2} \, dx \, dy$

let $u = xy^2$
$\Rightarrow du = 2xy \, dy$
$\frac{1}{2} \int e^u du$

$= \int_0^{\ln(3)} \int_0^1 xye^{xy^2} dy \, dx$

$= \int_0^{\ln(3)} \frac{1}{2} e^{xy^2} \Big|_{y=0}^{y=1} dx$

$= \int_0^{\ln(3)} \frac{1}{2}e^x - \frac{1}{2} \, dx$

$= \frac{1}{2}e^x - \frac{1}{2}x \Big|_0^{\ln(3)}$

$= \frac{1}{2}e^{\ln(3)} - \frac{1}{2}\ln(3) - \frac{1}{2} + 0$

$= \boxed{1 - \frac{1}{2}\ln(3)}$

11. (10 points each) Compute the sums of the following infinite series.

(a) $\sum_{n=2}^{\infty} e^{3-2n}$ $= e^3 \sum_{n=2}^{\infty} \left(\frac{1}{e^2}\right)^n$ ← geometric series with $r = 1/e^2 < 1$

$= e^3 \left(\frac{1/e^4}{1 - 1/e^2} \right)$

$= \frac{1/e}{\frac{e^2-1}{e^2}}$

$= \boxed{\frac{e}{e^2-1}}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{6^{2n+1}(2n+1)!}$ $= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \left(\frac{\pi}{6}\right)^{2n+1}$

Since $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$

$\Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \left(\frac{\pi}{6}\right)^{2n+1} = \sin(\pi/6) = \boxed{1/2}$

12. (10 points) The letter k is an arbitrary real number that has been fixed ahead of time. Show that the infinite series $\sum_{n=1}^{\infty} n^k 3^{-n}$ converges no matter what value of k has been chosen.

Ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^k}{3^{n+1}} \cdot \frac{3^n}{n^k} \right|$

$= \lim_{n\to\infty} \frac{1}{3} \cdot \frac{(n+1)^k}{n^k}$

$= 1/3 < 1$

So $\sum_{n=1}^{\infty} n^k 3^{-n}$ converges by the Ratio test

13. (10 points) Determine whether the following infinite series are convergent, or divergent. State which test(s) you use to reach your conclusion. Show all work.

(a) $\sum_{n=2}^{\infty} \frac{n^3}{\sqrt{n^4 - 2n^2 + 1}}$

$$\lim_{n\to\infty} \frac{n^3}{\sqrt{n^4-2n^2+1}} \approx \lim_{n\to\infty} \frac{n^3}{n^2} = \infty .$$

so diverges by the test for divergence

(b) $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2}$

$\arctan(n) \leq \pi/2$ for all $n \geq 0$

$\Rightarrow a_n = \frac{\arctan(n)}{n^2} \leq \frac{\pi/2}{n^2} = b_n$

Since $\sum b_n = \frac{\pi}{2} \sum \frac{1}{n^2}$ converges by the p-test, then $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2}$ converges by comparison.

14. (5 points each) Complete the following **definitions**.

(a) The infinite series $\sum_{n=1}^{\infty} a_n$ is **convergent** if

$\lim_{n\to\infty} S_n$ exists, where $S_n = a_1 + a_2 + \cdots + a_n$

(b) The infinite series $\sum_{n=1}^{\infty} a_n$ is **divergent** if

$\lim_{n\to\infty} S_n$ does not exist

(c) The infinite series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if

$\sum_{n=1}^{\infty} |a_n|$ converges

(d) The infinite series $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if

$\sum_{n=1}^{\infty} a_n$ converges, but not absolutely

15. (15 points) Determine whether the following series is conditionally convergent, absolutely convergent, or divergent. State which test(s) you use to reach your conclusion. Show all work.

$$\sum_{n=241}^{\infty} \frac{(-1)^{n+1}}{n\ln(n)}$$

Absolute? $\sum_{n=241}^{\infty} \frac{1}{n\ln(n)}$ let $f(x) = \frac{1}{x\ln(x)}$ ← positive, continuous, decreasing

$\lim_{t\to\infty} \int_{241}^{t} \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty} \ln(\ln(x))\Big|_{241}^{t} = \lim_{t\to\infty} \ln(\ln(t)) - \ln(\ln(241)) = \infty$

$\Rightarrow \sum_{n=241}^{\infty} \frac{1}{n\ln(n)}$ diverges by the integral test

Conditional? $a_n = \frac{1}{n\ln(n)}$ · $a_{n+1} \leq a_n$ ✓ · $\lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$ ✓

$\Rightarrow \sum_{n=241}^{\infty} \frac{(-1)^{n+1}}{n\ln(n)}$ converges by the alternating series test so converges conditionally

16. (10 points) Find the interval and radius of convergence of the following power series:

$$\sum_{n=12}^{\infty} e^n(x-2)^n$$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^{n+1}(x-2)^{n+1}}{e^n(x-2)^n}\right|$

$= e|x-2| < 1$

$\Rightarrow |x-2| < 1/e$ $\Rightarrow -1/e < x-2 < 1/e$ $\Rightarrow 2-1/e < x < 2+1/e$

Radius of conv. $= 1/e$

Check endpoints:

$x = 2-1/e$: $\sum_{n=12}^{\infty} e^n(-1/e)^n = \sum_{n=12}^{\infty}(-1)^n$ diverges by test for divergence

$x = 2+1/e$: $\sum_{n=12}^{\infty} e^n(1/e)^n = \sum_{n=12}^{\infty} 1^n$ diverges by the test for divergence

$\Rightarrow$ $\boxed{R = 1/e \text{ \& interval of convergence: } (2-1/e, 2+1/e)}$

17. (10 points each) Find Taylor series centered at $a = 0$ for the following functions. Simplify your answer. State the radius of convergence.

(a) $f(x) = \frac{x}{4-2x^3} = \frac{x}{4(1-\frac{2}{4}x^3)} = \frac{1}{4}\cdot\frac{x}{(1-\frac{1}{2}x^3)}$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \quad -1<x<1$

$\Rightarrow \frac{1}{1-\frac{1}{2}x^3} = \sum_{n=0}^{\infty} \left(\frac{1}{2}x^3\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n x^{3n} \quad , \quad -1 < \frac{1}{2}x^3 < 1 \Rightarrow \sqrt[3]{-2} < x < \sqrt[3]{2}$

$\Rightarrow \frac{1}{4}\frac{x}{1-\frac{1}{2}x^3} = \sum_{n=0}^{\infty} \frac{1}{4}\cdot\left(\frac{1}{2}\right)^n x^{3n+1} \quad , \quad \sqrt[3]{-2} < x < \sqrt[3]{2}$

$= \boxed{\sum_{n=0}^{\infty} \frac{1}{2^{n+2}} x^{3n+1}} \quad , \quad \boxed{R = \sqrt[3]{2}}$

(b) $f(x) = (1+2x)^{-2} = \frac{1}{(1+2x)^2}$

$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$

$= \sum_{n=0}^{\infty} n x^{n-1} \quad , \quad -1<x<1$

$\Rightarrow \frac{1}{(1+2x)^2} = \sum_{n=0}^{\infty} n(-2x)^{n-1} = \boxed{\sum_{n=0}^{\infty} (-1)^{n-1} 2^{n-1} n x^{n-1}} \quad , \quad -1 < -2x < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

$\Rightarrow \boxed{R = 1/2}$

18. (15 points) Find the degree three Taylor polynomial $T_3(x)$ at $a = 4$ for $f(x) = \sqrt{x}$.

Derivatives

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2} x^{-1/2}$

$f''(x) = -\frac{1}{4} x^{-3/2}$

$f'''(x) = \frac{3}{8} x^{-5/2}$

Evaluate at 4

$f(4) = 2$

$f'(4) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$f''(4) = -\frac{1}{4} \cdot \frac{1}{2^3} = \frac{-1}{32}$

$f'''(4) = \frac{3}{8} \cdot \frac{1}{2^5} = \frac{3}{8} \cdot \frac{1}{32}$

Coefficients

$a_0 = 2$

$a_1 = 1/4$

$a_2 = \frac{-1/32}{2} = -1/64$

$a_3 = \frac{\frac{3}{8} \cdot \frac{1}{32}}{6} = \frac{1}{16} \cdot \frac{1}{32} = \frac{1}{512}$

$$\boxed{T_3(x) = 2 + 1/4(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3}$$

$$\begin{array}{r} 32 \\ \times\ 16 \\ \hline 192 \\ 320 \\ \hline 512 \end{array}$$

19. Write out and sign the Honor Pledge.