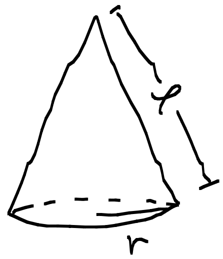
Lecture 9 - Surface Area

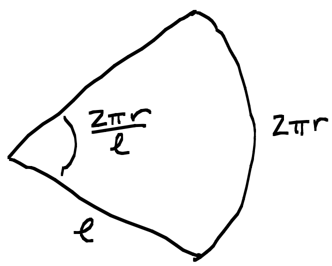
Note Title

Very straightforward from our discussion of arc length:

Given a cone:

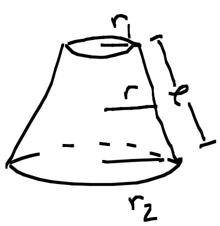


Can flatten to get a sector of a circle



$\Rightarrow$ Area $= \left(\frac{2\pi r}{\ell}\right) \cdot \frac{1}{2}\ell^2 = \pi r \ell$

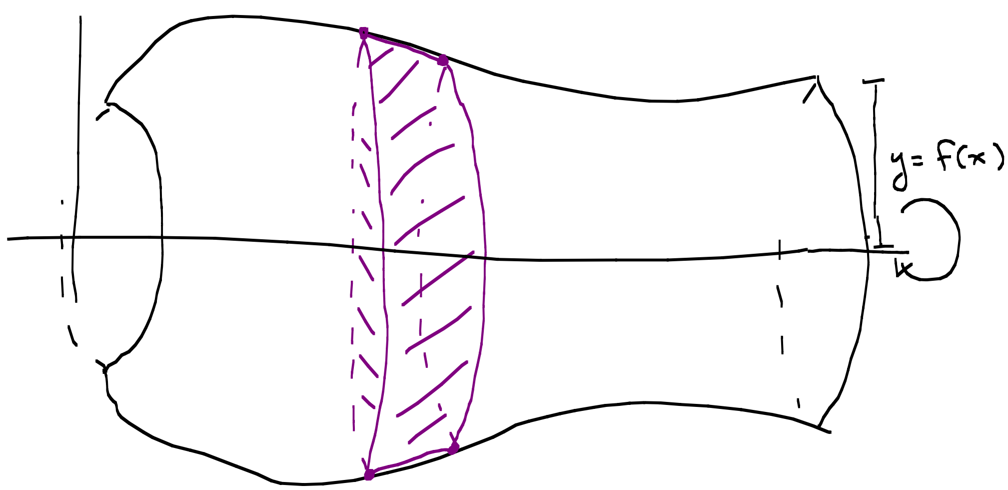
Could even do a truncated cone:



& surface area is $2\pi r \ell$, where
$r = \frac{1}{2}(r_1 + r_2) =$ "average radius"

Surface area follows from this: Over very small intervals, our curve looks like a line segment

$\Rightarrow$ The surface area of the tiny strip is $2\pi r \ell \approx$ area of section of surface



In this case, $r = y = f(x)$, and ℓ = arc length of the segment
$= \sqrt{1+(f'(x))^2}\,dx$.

Adding all up gives

The surface area of the surface of revolution formed by revolving $y = f(x)$ around the x-axis is

$$S = \int_a^b 2\pi \cdot f(x)\sqrt{1+(f'(x))^2}\,dx$$

Ex: $f(x) = \sqrt{9-x^2}$ between -2 & 2:

$$f'(x) = \frac{1}{\cancel{2}}(9-x^2)^{-1/2}\cdot(-\cancel{2}x) = \frac{-x}{\sqrt{9-x^2}}$$

$$\Rightarrow (f'(x))^2 = \frac{x^2}{9-x^2} \quad \& \quad 1+(f'(x))^2 = \frac{9-x^2}{9-x^2} + \frac{x^2}{9-x^2} = \frac{9}{9-x^2}$$

So $$S = \int_{-2}^{2} 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx = \int_{-2}^{2} 2\pi\sqrt{9-x^2}\cdot\sqrt{\frac{9}{9-x^2}}\,dx$$

$$= \int_{-2}^{2} 2\pi\cdot\sqrt{9}\,dx = 6\pi\, x\Big|_{-2}^{2} = \boxed{24\pi}$$

Have lots of ways to write this, since $\sqrt{1+(f'(x))^2}\,dx = ds$ had other names:

$$S = \int_a^b 2\pi\cdot y\,ds$$

This is an easier & more flexible form

y = distance from x-axis = distance from axis of revolution

ds = "element of arc length" $= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

If $x = g(y)$, then still get

$$S = \int_a^b 2\pi y\cdot ds = \int_a^b 2\pi y\sqrt{1+(g'(y))^2}\,dy$$

Ex $x = 2y^2$, revolved around x-axis $\quad 2 \le x \le 8 \Leftrightarrow 1 \le y \le 2$

$$S = \int_1^2 2\pi y \cdot \sqrt{1 + (4y)^2}\, dx$$

$u = 16y^2 + 1 \qquad y = 1 \Rightarrow u = 17$

$du = 32\, y\, dy \qquad y = 2 \Rightarrow u = 65$

$$= \frac{\pi}{16} \int_{17}^{65} \sqrt{u}\, du$$

$$= \frac{\pi}{16}\left(\frac{2}{3} u^{3/2}\right)\Big|_{17}^{65} = \boxed{\frac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right)}$$

Can also rotate about y-axis. Same ideas.

Axis of rotation	surface area
x-axis	$\int 2\pi y\, ds$
y-axis	$\int 2\pi x\, ds$

Ex: $x = \cos y$ between $y = 0$ and $y = \pi/4$

$x' = -\sin y$

$$S = \int_0^{\pi/4} 2\pi x \sqrt{1 + (-\sin y)^2}\, dy = \int_0^{\pi/4} 2\pi \cos y \sqrt{1 + \sin^2 y}\, dy$$

$u = \sin y \qquad y = 0 \Rightarrow u = 0$

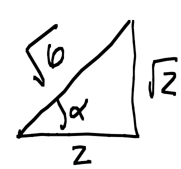
$du = \cos y\, dy \qquad y = \pi/4 \Rightarrow u = \sqrt{2}/2 \quad \Rightarrow \int_0^{\sqrt{2}/2} 2\pi \sqrt{1 + u^2}\, du$

$u = \tan t \qquad u = 0 \Rightarrow t = 0$

$du = \sec^2 t\, dt \qquad u = \sqrt{2}/2 \Rightarrow t = \tan^{-1}\left(\frac{\sqrt{2}}{2}\right) = \alpha \quad \Rightarrow \int_0^{\alpha} 2\pi \sec^3 t\, dt$

$$= \cancel{2}\pi \cdot \frac{1}{\cancel{2}}\left(\sec t \tan t + \ln|\sec t + \tan t|\right)\Big|_0^{\alpha}$$

@ 0: $\pi\left(1 \cdot 0 + \ln(1 + 0)\right) = 0 \qquad$ @ α: $\tan \alpha = \frac{\sqrt{2}}{2}$



$\Rightarrow \sec \alpha = \frac{\sqrt{6}}{2}$ & $S = \pi\left(\frac{\sqrt{6}}{2} \cdot \frac{\sqrt{2}}{2} + \ln\left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}\right)\right)$

$$S = \boxed{\pi\left(\frac{\sqrt{3}}{2} + \ln\left(\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}\right)\right)}$$