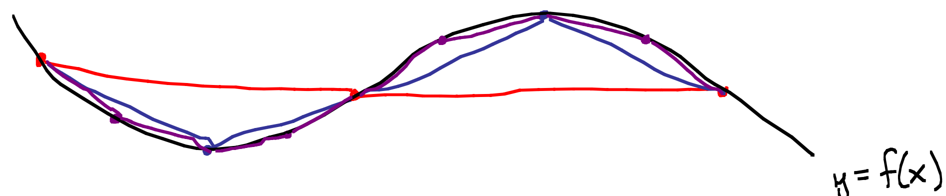
Lecture 8 – Arc Length

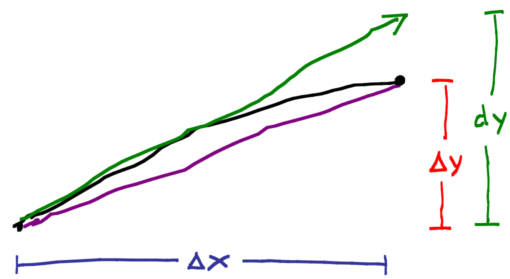
Note Title

We can approximate curves by line segments:



As the distance between sample points gets smaller (ie as Δx shrinks), the approximation gets better.

Blown-Up piece:



length of segment is $\sqrt{(\Delta x)^2+(\Delta y)^2}$

$= \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\cdot \Delta x$

Recall from Calc I: Δy hard to find, but if Δx is very small,

$$\Delta y \approx dy = f'(x)\,dx$$

or $$\frac{\Delta y}{\Delta x} \approx f'(x)$$

So the length of each segment is approximately

$$\sqrt{1+(f'(x))^2}\,dx$$

$\Rightarrow$ The arc length of $y=f(x)$ between $x=a$ and $x=b$ is

$$\int_a^b \sqrt{1+(f'(x))^2}\,dx.$$

<u>Ex</u>: $y=2x \Rightarrow y'=2$ so arc length between $x=0$ and $x=1$ is

$$\int_0^1 \sqrt{1+4}\,dx = \boxed{\sqrt{5}}$$

Ex: $y = \ln(\sec(x)) \Rightarrow y' = \tan(x)$ & arc length between $x=0$ & $\pi/4$ is

$$\int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/4} \sec x \, dx = \ln|\sec x + \tan x| \Big|_0^{\pi/4}$$

$$= \ln(\sqrt{2}+1) - \ln(1+0) = \boxed{\ln(1+\sqrt{2})}.$$

Ex: Set up the integral for the arc length between 0 and π of $y = \sin x$.

$y' = \cos x \Rightarrow$

$$\text{arc length} = \int_0^{\pi} \sqrt{1+(\cos x)^2}\, dx$$

Ex: Find the arc length of $y = \frac{x^2}{2}$ between $x=0$ and 1

$y' = x \Rightarrow$

$$\text{arc length} = \int_0^1 \sqrt{1+x^2}\, dx$$

$x = \tan t$
$dx = \sec^2 t \, dt$
$1+x^2 = \sec^2 t \, dt$

$x=0 \Rightarrow t=0$
$x=1 \Rightarrow t=\pi/4$

$$= \int_0^{\pi/4} \sec^3 t \, dt$$

Parts: $u = \sec t \Rightarrow du = \sec t \tan t \, dt$
$dv = \sec^2 t \Rightarrow v = \tan t$

$$= (\sec t \tan t)\Big|_0^{\pi/4} - \int_0^{\pi/4} \sec t \tan^2 t \, dt$$

$$= (\sqrt{2} - 0) - \int_0^{\pi/4} \sec t (\sec^2 t - 1)\, dt = \sqrt{2} - \int_0^{\pi/4} \sec^3 t \, dt + \int_0^{\pi/4} \sec t \, dt$$

$$= \sqrt{2} + (\ln(\sqrt{2}+1) - \ln(1)) - \int_0^{\pi/4} \sec^3 t \, dt$$

same as left-hand side!

$$\Rightarrow \text{arc length} = \boxed{\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1)}$$

Sometimes more useful to use y instead of x.

If $x = g(y)$, then the arc length between $y=a$ and b is

$$\int_a^b \sqrt{(g'(y))^2 + 1}\, dy$$

Ex: Consider the curve $4y^3 = 9x^2$ & let's find the arc length between $(0,0)$ and $(2/3, 1)$

$$x = \frac{2}{3} y^{3/2} \Rightarrow x' = y^{1/2}$$

So arc length $= \int_0^1 \sqrt{1 + (y^{1/2})^2}\, dy = \int_0^1 \sqrt{1+y}\, dy = \frac{2}{3}(1+y)^{3/2}\Big]_0^1$

$$= \frac{2}{3} \cdot 2^{3/2} - \frac{2}{3} = \boxed{\frac{4\sqrt{2} - 2}{3}}$$

Arc length itself is a function:

$$s(x) = \int_a^x \sqrt{1 + (f'(x))^2}\, dx \qquad \text{if } y = f(x)$$

or

$$s(y) = \int_a^y \sqrt{(g'(y))^2 + 1}\, dy \qquad \text{if } x = g(y)$$

The fundamental theorem of calculus says what s' is:

$$\frac{ds}{dx} = \sqrt{1 + (f'(x))^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

or

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{“}=\text{”} \quad \sqrt{(dx)^2 + (dy)^2}$$ ← we'd get this if we started w/ $s(y)$.

Use short-hand: $(ds)^2 = (dx)^2 + (dy)^2$

Ex: $x = \frac{2}{3} y^{3/2}$. Then between $(0,0)$ and $(2/3\, y^{3/2}, y)$, arc length is

$$s(y) = \int_0^y \sqrt{y+1}\, dy = \frac{2}{3}(y+1)^{3/2}\Big|_0^y$$

$$= \boxed{\frac{2}{3}(y+1)^{3/2} - \frac{2}{3}}$$ ← So we could plug-in any y we want here.