Lecture 8 - Arc Length

We can approximate curves by line segments:

As the distance between sample points gets smaller (i.e., as $\Delta x$ shrinks), the approximation gets better.

Blown-up piece:

Recall from Calc I: $\Delta y$ hard to find, but if $\Delta x$ is very small,

$$\Delta y \approx dy = f'(x) \Delta x$$

or

$$\frac{\Delta y}{\Delta x} \approx f'(x)$$

So the length of each segment is approximately

$$\sqrt{1 + (f'(x))^2} \, dx$$

⇒ The arc length of $y = f(x)$ between $x = a$ and $x = b$ is

$$\int_a^b \sqrt{1 + (f'(x))^2} \, dx.$$ 

Ex: $y = 2x \Rightarrow y' = 2$, so arc length between $x = 0$ and $x = 1$ is

$$\int_0^1 \sqrt{1 + 4} \, dx = \sqrt{5}.$$
Ex: \( y = \ln(\sec(x)) \Rightarrow y' = \tan(x) \) & arc length between \( x = 0 \) & \( \frac{\pi}{4} \) is

\[
\int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x} \, dx = \int_0^{\frac{\pi}{4}} \sec x \, dx = \ln |\sec x + \tan x| \bigg|_0^{\frac{\pi}{4}}
\]

\[
= \ln \left( \sqrt{2} + 1 \right) - \ln (1 + 0) = \left[ \ln (1 + \sqrt{2}) \right].
\]

Ex: Set up the integral for the arc length between of \( y = \sin x \),

\[
y' = \cos x \Rightarrow
\]

arc length \( = \int_0^{\frac{\pi}{2}} \sqrt{1 + \cos^2 x} \, dx \)

Ex: Find the arc length of \( y = \frac{x^2}{2} \) between \( x = 0 \) and \( 1 \)

\[
y' = x \Rightarrow \int_0^1 \sqrt{1 + x^2} \, dx \quad \text{where} \quad x = \tan t \quad \text{and} \quad dx = \sec^2 t \, dt \quad \text{with} \quad t = 0 \rightarrow t = \frac{\pi}{4} \]

\[
= \int_0^{\frac{\pi}{4}} \sec^3 t \, dt
\]

Parts: \( u = \sec t \Rightarrow du = \sec t \tan t \, dt \)

\( dv = \sec^2 t \Rightarrow v = \tan t \)

\[
= (\sec t \tan t) \bigg|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec t \tan^2 t \, dt
\]

\[
= \left( \sqrt{2} - 0 \right) - \int_0^{\frac{\pi}{4}} \sec t (\sec^2 t - 1) \, dt = \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3 t \, dt + \int_0^{\frac{\pi}{4}} \sec t \, dt
\]

\[
= \sqrt{2} + \left( \ln (\sqrt{2} + 1) - \ln (1) \right) - \int_0^{\frac{\pi}{4}} \sec^3 t \, dt
\]

\( \Rightarrow \) arc length \( = \frac{\sqrt{2}}{2} + \frac{1}{2} \ln (\sqrt{2} + 1) \)
Sometimes more useful to use $y$ instead of $x$.

If $x = g(y)$, then the arc length between $y = a$ and $b$ is

$$
\int_{a}^{b} \sqrt{\left(g'(y)\right)^2 + 1} \, dy
$$

Ex: Consider the curve $y^3 = 9x^2$ & let's find the arc length between $(0,0)$ and $(2/3, 1)$

$$
x = \frac{2}{3} y^{3/2} \quad \Rightarrow \quad x' = y^{1/2}
$$

So arc length $= \int_{0}^{1} \sqrt{1 + (y^{1/2})^2} \, dy = \int_{0}^{1} \sqrt{1 + y} \, dy = \frac{2}{3} \left[ (y + 1)^{3/2} \right]_{0}^{1}
$$

$$
= \frac{2}{3} \cdot 2^{3/2} - \frac{2}{3} = \frac{4\sqrt{2} - 2}{3}
$$

Arc length itself is a function:

$$
s(x) = \int_{a}^{x} \sqrt{1 + (f'(x))^2} \, dx \quad \text{if} \quad y = f(x)
$$

or

$$
s(y) = \int_{a}^{y} \sqrt{g'(y)^2 + 1} \, dy \quad \text{if} \quad x = g(y)
$$

The fundamental theorem of calculus says what $s'$ is:

$$
\frac{ds}{dx} = \sqrt{1 + (f'(x))^2} = \sqrt{1 + (\frac{dy}{dx})^2}
$$

or

$$
ds = \sqrt{1 + (\frac{dy}{dx})^2} \, dx = \sqrt{(dx)^2 + (dy)^2}
$$

Use shorthand: $(ds)^2 = (dx)^2 + (dy)^2$

Ex: $x = \frac{2}{3} y^{3/2}$. Then between $(0,0)$ and $(2/3, y^{3/2}, y)$, arc length is

$$
s(y) = \int_{0}^{y} \sqrt{y + 1} \, dy = \frac{2}{3} \left[ (y + 1)^{3/2} \right]_{0}^{y}
$$

$$
= \frac{2}{3} \left( (y + 1)^{3/2} - \frac{2}{3} \right) \quad \text{so we could plug-in any $y$ we want here.}