Lecture 7 - Improper Integrals

Note Title

Today we'll cover 2 cases in definite integration:

a) Integrals w/ infinite limits

b) Integrals w/ infinite discontinuities in the integrand (ie vertical asymptotes)

I. Def • If $\int_a^t f(x)\,dx$ exists for at $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

Provided these limits exist

• If $\int_t^b f(x)\,dx$ exists for all $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to\infty} \int_t^b f(x)\,dx$$

• The improper integral $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$ is <u>convergent</u> if the limit exists and <u>divergent</u> otherwise.

<u>Ex</u>
$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2}\right) = \frac{1}{2}.$$

So this improper integral is convergent.

<u>Ex</u>
$$\int_{-\infty}^1 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{t\to-\infty}\int_t^1 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{t\to-\infty}\left(\frac{3}{2}x^{2/3}\Big]_t^1\right) =$$

$$\lim_{t\to-\infty}\left(\frac{3}{2}-\frac{3}{2}t^{2/3}\right) = -\infty.$$

So this improper integral is divergent.

Can also handle doubly infinite integrals.

<u>Def</u> If $\int_{-\infty}^c f(x)\,dx$ and $\int_c^\infty f(x)\,dx$ are convergent, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx.$$

We can choose whatever we want for c in the above.

This could be very different from $\lim_{t \to \infty} \int_{-t}^{t} f(x)\,dx$

<u>Ex</u>: $\int_{-\infty}^{\infty} \sin(x)\,dx = \int_{0}^{\infty} \sin(x)\,dx + \int_{-\infty}^{0} \sin(x)\,dx$

$$\lim_{t \to \infty} \int_0^t \sin x\,dx + \lim_{s \to -\infty} \int_s^0 \sin x\,dx$$

$$\lim_{t \to \infty} \left(1 - \cos t\right) + \lim_{s \to -\infty} \left(\cos s - 1\right)$$

$$\text{DNE} \qquad\qquad \text{DNE.}$$

But $\lim_{t \to \infty} \int_{-t}^{t} \sin(x)\,dx = \lim_{t \to \infty} \left(\cos(-t) - \cos t\right) = \lim_{t \to \infty} (0) = 0.$

<u>Ex</u> $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{s \to -\infty} \int_s^1 \frac{dx}{1+x^2} + \lim_{t \to \infty} \int_1^t \frac{dx}{1+x^2}$

$$\lim_{s \to -\infty} \left(\arctan x \Big|_s^1\right) + \lim_{t \to \infty} \left(\arctan(x) \Big|_1^t\right)$$

$$\lim_{s \to -\infty} \left(\pi/4 - \arctan(s)\right) + \lim_{t \to \infty} \left(\arctan(t) - \pi/4\right)$$

$$3\pi/4 + \pi/4 = \boxed{\pi}$$

So this integral converges.

II. If f has a vertical asymptote at a point, we do something similar:

<u>Def</u> •) If f is continuous on $[a,b)$, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

•) If f is continuous on $(a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

If the limits exist, we again say the integral is convergent.

Ex: $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+} \left(2\sqrt{x} \Big|_t^1 \right) =$

bad at 0

$$\lim_{t\to 0^+} \left(2 - 2\sqrt{t}\right) = \boxed{2}.$$

Ex: $\displaystyle\int_{-1}^0 \frac{dx}{x} = \lim_{t\to 0^-} \int_{-1}^t \frac{dx}{x} = \lim_{t\to 0^-} \left(\ln|t| - \ln|-1| \right) = \lim_{t\to 0^-} \ln|t| = -\infty$

This integral is divergent.

If the asymptote is in the middle, we must break up the integral.

Def If f has a vertical asymptote at c, $a<c<b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

evaluate as above

If there are multiple bad points, break up the integral along each.

Ex: $\displaystyle\int_{-1}^1 \frac{dx}{x^2}$ $\quad \frac{1}{x^2}$ bad at $x=0$, so

$$\int_{-1}^1 \frac{dx}{x^2} = \int_{-1}^0 \frac{dx}{x^2} + \int_0^1 \frac{dx}{x^2} = \lim_{t\to 0^-} \int_{-1}^t \frac{dx}{x^2} + \lim_{s\to 0^+} \int_s^1 \frac{dx}{x^2}$$

$$= \lim_{t\to 0^-} \left(-\frac{1}{x} \Big|_{-1}^t \right) + \lim_{s\to 0^+} \left(-\frac{1}{x} \Big|_s^1 \right)$$

$$= \underbrace{\lim_{t\to 0^-} \left(-\frac{1}{t} - 1 \right)}_{\text{DNE } (\infty)} + \underbrace{\lim_{s\to 0^+} \left(-1 + 1/s \right)}_{\text{DNE } (\infty)}$$

So this integral is divergent, even though $-\frac{1}{x}\Big|_{-1}^1 = -2$

Can often tell convergence or divergence by comparison.

Thm : Given $f(x)$ and $g(x)$, both ≥ 0, f defined and cont $\leftrightarrow$ g defined and cont

•) If $0 \leq f(x) \leq g(x)$, and $\int_a^b g(x)\,dx$ converges, then

$$\int_a^b f(x)\,dx \text{ converges.}$$

•) If $0 \leq g(x) \leq f(x)$ and $\int_a^b g(x)\,dx$ diverges, then

$$\int_a^b f(x)\,dx \text{ diverges.}$$

Ex : $\int_1^\infty \frac{1}{x^3}\,dx$ converges. For all $x \geq 1$, $\frac{1}{x^3} \geq \frac{1}{x^3+1} \Rightarrow$

$$\int_1^\infty \frac{1}{x^3+1}\,dx \text{ converges.}$$

Ex : $\int_1^\infty \frac{1}{x}\,dx$ diverges. For all $x \geq 1$, $\frac{1}{x} \leq \frac{1}{\sqrt{x}}$, so

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx \text{ diverges.}$$