

Lecture 6 – Integration Techniques

Note Title

Rationalization:

One last case: $\sqrt[n]{ax+b}$

Do a "rationalizing" u-sub: $u = \sqrt[n]{ax+b}$

$$du = \frac{a}{n} \cdot (ax+b)^{\frac{1}{n}-1} dx = \frac{a}{n} \cdot u^{\frac{1}{n}-1} dx$$

$$\frac{n}{a} u^{n-1} du = dx$$

$$u = x^{\frac{1}{3}}$$

$$du = \frac{1}{3} x^{-\frac{2}{3}} dx = \frac{1}{3} (x^{\frac{1}{3}})^{-2} dx = \frac{1}{3} u^{-2} dx$$

$$3u^2 du = dx$$

$$= 3u^2 e^u - 6u e^u + 6e^u + C$$

$$= \boxed{3x^{\frac{2}{3}} e^{\frac{3x}{2}} - 6x^{\frac{1}{3}} e^{\frac{3x}{2}} + 6e^{\frac{3x}{2}} + C}$$

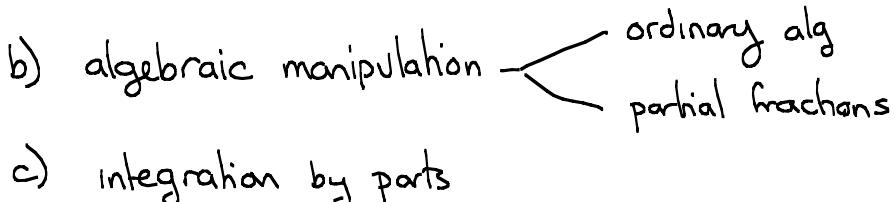
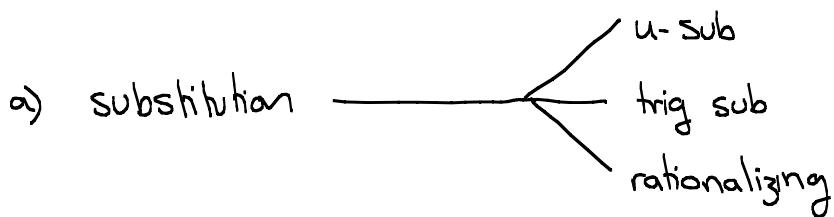
$$\begin{array}{rcl} 3u^2 & e^u & - \\ 6u & e^u & + \\ 6 & e^u & - \\ 0 & e^u & + \end{array}$$

This process usually results in something that is

- a) a less complicated integrand
- b) usually integrated by parts

How do we integrate?

Seen 3 families of techniques:



c) integration by parts

We might have to try multiple things.

I. Try to rearrange.

II. Look for u-subs

III. Try method based on type: trig int, rational function, radicals, trig sub

IV. Repeat w/ trickier subs & parts.

$$\underline{\text{Ex}}: \int \frac{\sin^3 x}{\cos x} dx = \int \sin x \left(\frac{1 - \cos^2 x}{\cos x} \right) dx \quad u = \cos x \quad du = -\sin x dx$$

$$= \int \frac{u^2 - 1}{u} du = \frac{u^2}{2} - \ln u = \boxed{\frac{\cos^2 x}{2} - \ln(\cos x) + C}$$

$$\underline{\text{Ex}} \quad \int \frac{x}{\sqrt{4-x^4}} dx \quad \begin{aligned} \text{See } x dx, \text{ so try } u = x^2, du = 2x dx \\ u = x^2 \Rightarrow x^4 = u^2 \end{aligned}$$

$$\begin{aligned} &= \int \frac{1}{2} \frac{du}{\sqrt{4-u^2}} \quad \begin{aligned} \text{trig sub: } u &= 2 \sin \theta \\ du &= 2 \cos \theta d\theta \end{aligned} \\ &= \int \frac{1}{2} \frac{2 \cos \theta}{2 \cos \theta} d\theta = \int \frac{1}{2} d\theta = \frac{1}{2} \theta = \frac{1}{2} \sin^{-1}(u/2) = \boxed{\frac{1}{2} \sin^{-1}(x^2/2) + C} \end{aligned}$$

$$\underline{\text{Ex}}: \int_{-1}^1 \frac{e^{\arctan(x)}}{1+x^2} dx \quad \begin{aligned} \text{can't handle, so sub away} \\ u = \tan^{-1} x \quad du = \frac{1}{1+x^2} dx \end{aligned}$$

$$\int_{-\pi/4}^{\pi/4} e^u du = \quad \begin{aligned} x = -1 \Rightarrow u = -\pi/4 \\ x = 1 \Rightarrow u = \pi/4 \end{aligned}$$

$$\boxed{e^{\pi/4} - e^{-\pi/4}}$$

$$\underline{\text{Ex}}: \int \frac{e^{2x}}{1+e^{4x}} dx \quad \begin{aligned} 2 \text{ choices: } v &= 2x \Rightarrow \int \frac{1}{2} \frac{e^v}{1+e^{2v}} dv \\ \text{or} \end{aligned}$$

$$u = \frac{e^{2x}}{2e^{2x}} \Rightarrow \int \frac{1}{2} \frac{1}{1+u^2} du$$

$$\text{So get } \frac{1}{2} \tan^{-1}(u) + C = \boxed{\frac{1}{2} \tan^{-1}(e^{2x}) + C}$$

$$\underline{\text{Ex}}: \int e^{x+e^x} dx = \int e^x e^{e^x} dx \stackrel{\substack{\uparrow \\ \text{algebra}}}{=} \int e^u du = e^u + C = \boxed{e^{e^x} + C}$$

$u = e^x$
 $du = e^x dx$

$$\underline{\text{Ex}} \quad \int x^5 e^{-x^3} dx \quad e^{-x^3} \text{ makes this tough to integrate, so}$$

$u = -x^3 \Rightarrow du = -3x^2 dx$

$$\int x^5 e^{-x^3} dx = \int x^3 \cdot e^{-x^3} \cdot x^2 dx = \int (-u) e^u \cdot -\frac{1}{3} du = \frac{1}{3} \int u e^u du$$

$$\begin{array}{rcl} u & \xrightarrow{\quad e^u \quad -} & = \frac{1}{3} (ue^u - e^u) + C \\ 1 & \xrightarrow{\quad e^u \quad +} & = \boxed{\frac{1}{3} (-x^3 e^{-x^3} - e^{-x^3}) + C} \\ 0 & \xrightarrow{\quad e^u \quad -} & \end{array}$$