Lecture 6 – Integration Techniques

Note Title

Rationalization:

One last case: $\sqrt[n]{ax+b}$

Do a "rationalizing" u-sub: $u = \sqrt[n]{ax+b}$

$$du = \frac{a}{n}\cdot(ax+b)^{\frac{1-n}{n}}\,dx = \frac{a}{n}\cdot u^{1-n}\,dx$$

$$\frac{n}{a}u^{n-1}\,du = dx$$

<u>Ex</u> $\int e^{\sqrt[3]{x}}\,dx$

$u = x^{1/3}$

$du = \frac{1}{3}x^{-2/3}\,dx = \frac{1}{3}(x^{1/3})^{-2}\,dx = \frac{1}{3}u^{-2}\,dx$

$3u^2\,du = dx$

$$= \int 3u^2 e^u\,du$$

$3u^2$	e^u	$-$
$6u$	e^u	$+$
6	e^u	$-$
0	e^u	$+$

$$= 3u^2e^u - 6ue^u + 6e^u + C$$

$$= \boxed{3x^{2/3}e^{\sqrt[3]{x}} - 6x^{1/3}e^{\sqrt[3]{x}} + 6e^{\sqrt[3]{x}} + C}$$

This process usually results in something that is

a) a less complicated integrand

b) usually integrated by parts

How do we integrate?

Seen 3 families of techniques:

a) substitution
- u-sub
- trig sub
- rationalizing

b) algebraic manipulation
- ordinary alg
- partial fractions

c) integration by parts

We might have to try multiple things.

I. Try to rearrange.

II. Look for u-subs

III. Try method based on type: trig int, rational function, radicals, trig sub

IV. Repeat w/ trickier subs & parts.

Ex: $\int \frac{\sin^3 x}{\cos x}\,dx = \int \sin x \left(\frac{1-\cos^2 x}{\cos x}\right) dx$ $\quad u = \cos x \quad du = -\sin x\,dx$

$= \int \frac{u^2-1}{u}\,du = \frac{u^2}{2} - \ln u = \boxed{\frac{\cos^2 x}{2} - \ln(\cos x) + C}$

Ex $\int \frac{x}{\sqrt{4-x^4}}\,dx$ $\quad$ see $x\,dx$, so try $u = x^2$, $du = 2x\,dx$

$u = x^2 \Rightarrow x^4 = u^2$

$= \int \frac{1}{2} \frac{du}{\sqrt{4-u^2}}$ $\quad$ trig sub: $\quad u = 2\sin\theta$

$du = 2\cos\theta\,d\theta$

$= \int \frac{1}{2} \frac{2\cos\theta\,d\theta}{2\cos\theta} = \int \frac{1}{2}\,d\theta = \frac{1}{2}\theta = \frac{1}{2}\sin^{-1}(u/2) = \boxed{\frac{1}{2}\sin^{-1}(x^2/2) + C}$

Ex: $\int_{-1}^{1} \frac{e^{\arctan(x)}}{1+x^2}\,dx$ $\quad$ can't handle, so sub away

$u = \tan^{-1} x \qquad du = \frac{1}{1+x^2}\,dx$

$x = -1 \Rightarrow u = -\pi/4$

$x = 1 \Rightarrow u = \pi/4$

$\int_{-\pi/4}^{\pi/4} e^u\,du =$

$\boxed{e^{\pi/4} - e^{-\pi/4}}$

Ex: $\int \frac{e^{2x}}{1+e^{4x}}\,dx$ $\quad$ 2 choices: $\quad v = 2x \Rightarrow \int \frac{1}{2} \frac{e^v}{1+e^{2v}}\,dv$

$\Downarrow u = e^v$

or

$u = e^{2x}$, $du = 2e^{2x}$ $\Rightarrow \int \frac{1}{2} \frac{1}{1+u^2}\,du$

So get $\frac{1}{2}\tan^{-1}(u) + C = \boxed{\frac{1}{2}\tan^{-1}(e^{2x}) + C}$

<u>Ex</u>: $\int e^{x+e^x}\,dx \underset{\substack{\uparrow \\ \text{algebra}}}{=} \int e^x e^{e^x}\,dx = \int e^u\,du = e^u + C = \boxed{e^{e^x} + C}$

$u = e^x$

$du = e^x\,dx$

<u>Ex</u> $\int x^5 e^{-x^3}\,dx$ $\qquad e^{-x^3}$ makes this tough to integrate, so

$u = -x^3 \Rightarrow du = -3x^2\,dx$

$$\int x^5 e^{-x^3}\,dx = \int x^3 \cdot e^{-x^3} \cdot x^2\,dx = \int (-u)e^u \cdot -\frac{1}{3}\,du = \frac{1}{3}\int ue^u\,du$$

u	e^u	$-$
1	e^u	$+$
0	e^u	1

$$= \frac{1}{3}(ue^u - e^u) + C$$

$$= \boxed{\frac{1}{3}(-x^3e^{-x^3} - e^{-x^3}) + C}$$