Lecture 5 - Partial Fractions

Note Title

Today we will continue studying rational functions:
functions like $\frac{7x}{3x^2+9}$ or $\frac{x^7}{9x^2+3}$ or $\frac{p(x)}{q(x)}$, p & q polynomials. Main methods: long division & partial fractions

General Idea: break up a rational function into simpler ones.

Ex: $\frac{x^2}{x+3}$ deg(num) $= 2 > 1 =$ deg(denom)
$\Rightarrow$ divide

$$\begin{array}{r} x - 3 \\ x+3 \overline{\smash{)} x^2 \quad\quad} \\ -\ \underline{x^2 + 3x} \\ -3x \quad \\ -\ \underline{-3x - 9} \\ 9 \end{array}$$

9 — remainder

So $\boxed{\frac{x^2}{x+3} = x-3+\frac{9}{x+3}}$

$$\int \frac{x^2}{x+3}\,dx = \int x-3+\frac{9}{x+3}\,dx = \boxed{\frac{x^2}{2} - 3x + 9\ln(x+3) + C}$$

Long division works whenever deg(num) $\geq$ deg(denom)
so in this case, use it!

Partial Fractions: factor the denom & break up according to the factors.

Ex: $\frac{1}{x-1} - \frac{1}{x} = \frac{x}{x(x-1)} - \frac{(x-1)}{x(x-1)}$
$= \frac{1}{x(x-1)}$

so we conclude:

$\frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}$ & $\int \frac{1}{x(x-1)}dx = \int \frac{-1}{x} + \frac{1}{x-1}dx = \boxed{\ln|x-1| - \ln|x| + C}$

2 kinds of factors: linear factors $ax+b$

irreducible quadratic factors ax^2+bx+c, $b^2-4ac<0$

for each, there are 2 possibilities:

simple factor the factor occurs only once
repeated factor " " " more than once.

$\Rightarrow$ 4 cases:

I. all simple linear factors $(x-r_1)\cdots(x-r_n)$ r_i all distinct

Write $\dfrac{p(x)}{(x-r_1)\cdots(x-r_n)} = \dfrac{A_1}{x-r_1} + \dots + \dfrac{A_n}{x-r_n}$

Ex $\dfrac{2x+5}{x^2-6x+5} = \dfrac{2x+5}{(x-5)(x-1)} = \dfrac{A}{x-5} + \dfrac{B}{x-1}$

$\Rightarrow$ $2x+5 = A(x-1) + B(x-5) = (A+B)x + (-A-5B)$

$\Rightarrow$ $\begin{array}{l} A+B=2 \\ -A-5B=5 \end{array}$ $\Rightarrow$ $-4B=7$ $\Rightarrow$ $B = -7/4$ $\Rightarrow$ $A = 15/4$

So $\dfrac{2x+5}{x^2-6x+5} = \boxed{\dfrac{15/4}{x-5} - \dfrac{7}{4}\cdot\dfrac{1}{x-1}}$

Aside: Easiest way to find coefs is to evaluate at <u>roots</u>:

$2x+5 = A\cdot(x-1) + B(x-5)$ @ $x=1$: $7 = A\cdot 0 + B(-4) = -4B$

@ $x=5$: $15 = A\cdot 4 + B\cdot 0 = 4A$

Harder to do in other cases

II. Simple quadratic factors: Same idea

Write $\dfrac{p(x)}{(x^2+b_1x+c_1)\cdots} = \dfrac{A_1x+B_1}{x^2+b_1x+c_1} + \dots$

& cross multiply

Ex $\dfrac{x}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

$\rightsquigarrow$ $x = A(x^2+1) + (Bx+C)(x-1)$

$= (A+B)x^2 + (C-B)x + (A-C)$

$\Rightarrow$ $\left.\begin{array}{l} A+B=0 \\ -B+C=1 \\ A-C=0 \end{array}\right\}$ $\begin{array}{l} A+C=1 \\ A-C=0 \end{array}$ $\Rightarrow$ $\begin{array}{l} A = 1/2,\ C = 1/2 \\ A+B = 0 \Rightarrow B = -1/2 \end{array}$

$\dfrac{x}{(x-1)(x^2+1)} = \dfrac{\left(\frac{1}{2}\right)}{x-1} + \dfrac{\left(\frac{-1}{2}x+\frac{1}{2}\right)}{x^2+1}$

III. Repeated factors:

$$\frac{p(x)}{(x-r_1)^{m_1}\cdots} = \frac{A_1}{(x-r_1)} + \frac{A_2}{(x-r_1)^2} + \cdots + \frac{A_{m_1}}{(x-r_1)^{m_1}} + \cdots$$

Ex:

$$\frac{1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$\Rightarrow \quad 1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

@ $x=1$: $\quad 1 = A\cdot 0\cdot 2 + B\cdot 2 + C\cdot 0 = 2B \quad \Rightarrow \quad B = 1/2$

@ $x=-1$: $\quad 1 = A\cdot(-2)\cdot 0 + B\cdot 0 + C\cdot(-2)^2 = 4C \quad \Rightarrow \quad C = 1/4$

@ $x=0$ $\quad 1 = A\cdot -1 + B\cdot 1 + C(-1)^2 = B + C - A = \frac{3}{4} - A \quad \Rightarrow \quad A = -1/4$

So $$\int \frac{1}{(x-1)^2(x+1)}\,dx = \int -\frac{1}{4}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{(x-1)^2} + \frac{1}{4}\cdot\frac{1}{x+1}\,dx =$$

$$\boxed{-\frac{1}{4}\ln(x-1) - \frac{1}{2}\cdot\frac{1}{x-1} + \frac{1}{4}\ln|x+1| + C}$$

IV. Last case: repeated quadratic factors

Same idea: $$\frac{p(x)}{(x^2+bx+c)^{r_1}} = \frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \cdots + \frac{A_{r_1}x+B_{r_1}}{(x^2+bx+c)^{r_1}}$$

Ex Find the form of

$$\frac{x^2+3x}{(x^2+x+1)^2(x+1)^2} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{E}{(x+1)} + \frac{F}{(x+1)^2}$$