# Lecture 4 – TRIG SUBSTITUTION

Note Title

The other cases are quite tricky. These include:

① powers of $\tan x$ w/o $\sec x$

Switch all $\tan^2 x$ to $\sec^2 x - 1$.

This often makes it an earlier case.

Ex $\int \tan^5 x \, dx = \int \tan x \left(\sec^2 x - 1\right)^2 dx = \int \tan x - 2\tan x \sec^2 x + \tan x \sec^4 x \, dx$

$$= \boxed{\ln|\sec x| - \tan^2 x + \frac{1}{4}\sec^4 x + C}$$

② even powers of $\tan x$ w/ odd powers of $\sec x$:

switch all $\tan^2 x$ to $\sec^2 x - 1$ & Integrate by parts

w/ $dv = \sec^2 x \, dx$

Did all this to integrate functions like $\sqrt{x^2 + a^2}$ : Trig Substitution

These are direct substitutions: if $x = g(t)$, then $dx = g'(t)\,dt$

Some examples:

Ex: $\int \frac{dx}{x^2 + 1}$ If $x = \tan t$, then $dx = \sec^2 t \, dt$

$$= \int \frac{\sec^2 t}{\tan^2 t + 1} dt = \int \frac{\sec^2 t}{\sec^2 t} dt = \int 1 \, dt = t + C$$

$x = \tan t \Rightarrow t = \tan^{-1} x \Rightarrow \; = \boxed{\arctan x + C}$

$= \arctan x$

What did we learn? If we see $x^2 + 1$, then substituting $x = \tan t$ can simplify!

Ex $\int \frac{dx}{\sqrt{x^2 + 1}}$ $\quad x = \tan t$

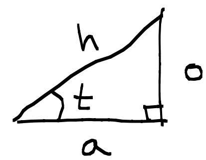
$dx = \sec^2 t \, dt$

$$= \int \frac{\sec^2 t}{\sec t}\, dt = \int \sec t \, dt = \ln\left|\sec t + \underset{\substack{\uparrow \\ x}}{\tan t}\right| + C$$

Now if $x = \tan t$, then what is $\sec t$?

① PT: $\sec t = \sqrt{\tan^2 t + 1} = \sqrt{x^2+1}$

② Draw a triangle:




$a$ = adjacent } side
$o$ = opposite }
$h$ = hypotinuse

Knowing 2 gives all 3

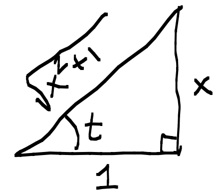
Trig functions are ratios:

$\sin t = \frac{o}{h}$ $\qquad \csc t = h/o$

$\cos t = \frac{a}{h}$ $\qquad \sec t = h/a$

$\tan t = \frac{o}{a}$ $\qquad \cot t = a/o$

So if $\tan t = x$, can say




$\sin t = \frac{x}{\sqrt{x^2+1}}$ $\qquad \csc t = \frac{\sqrt{x^2+1}}{x}$

$\cos t = \frac{1}{\sqrt{x^2+1}}$ $\qquad \sec t = \sqrt{x^2+1}$

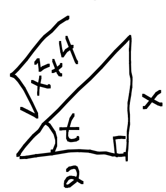
What about more complicated things?

Ex: $\int \frac{dx}{(x^2+4)^{3/2}}$ $\qquad$ If we try $x = \tan t$, get $\tan^2 t + 4$. :(

Instead, want coef of $\tan^2 t$ to match

$x = 2\tan t \quad \longleftrightarrow \quad x/2 = \tan t$

$dx = 2\sec^2 t \, dt$



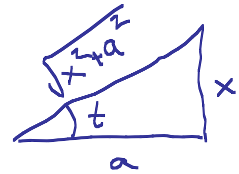

$$\int \frac{2\sec^2 t}{(4\sec^2 t)^{3/2}}\, dt = \frac{1}{4}\int \frac{\sec^2 t}{\sec^3 t}\, dt$$

$$= \frac{1}{4}\int \frac{1}{\sec t}\, dt = \frac{1}{4}\int \cos t \, dt = \frac{1}{4}\sin t + C$$

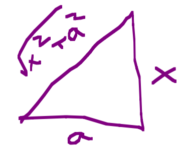
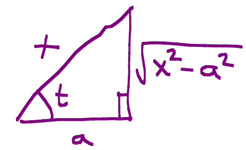
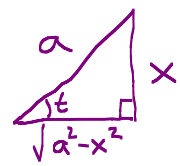
$$\tan t = x/2 \Rightarrow \sin t = x/\sqrt{x^2+4}$$

$$= \boxed{\frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C}$$

If we see $x^2+a^2$, try $x = a\tan t \Rightarrow$




This is part of a general pattern:

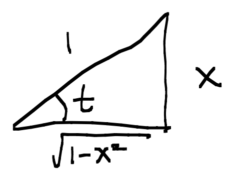
| | If we see | Then try | Triangle |
|---|---|---|---|
| $a^2\sec^2 t \leftarrow$ | $x^2+a^2$ | $x = a\tan t$ |  |
| $a^2\tan^2 t \leftarrow$ | $x^2-a^2$ | $x = a\sec t$ |  |
| $a^2\cos^2 t \leftarrow$ | $a^2-x^2$ | $x = a\sin t$ |  |





<u>Ex</u> $\displaystyle\int \frac{dx}{(1-x^2)^{3/2}}$

$x = \sin t$
$dx = \cos t\, dt$



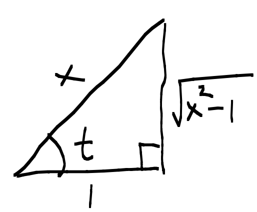

$$= \int \frac{\cos t}{(1-\sin^2 t)^{3/2}}\,dt = \int \frac{\cos t}{\cos^3 t}\,dt = \int \sec^2 t\,dt = \tan t + C$$

(from $\Delta$) $\quad \tan t = \frac{x}{\sqrt{1-x^2}} \Rightarrow \quad = \boxed{\frac{x}{\sqrt{1-x^2}} + C}$

<u>Ex</u> $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

$x = \sec t \Rightarrow x^2-1 = \sec^2 t - 1 = \tan^2 t$

$dx = \sec t \cdot \tan t$




$$= \int \frac{\sec t \cdot \tan t}{\sec t \cdot \sqrt{\tan^2 t}}\,dt = \int 1\,dt = t + c$$

$$= \boxed{\sec^{-1} x + C}$$

arcsec $x$

Definite Integration: When we change to $t$, change limits:

$x = a \tan t \Rightarrow t = \arctan(x/a)$ $\quad -\pi/2 < t < \pi/2$

$x = a \sec t \Rightarrow t = \operatorname{arcsec}(x/a)$ $\quad 0 < t < \pi/2$ or $\pi/2 < t < \pi$

$x = a \sin t \Rightarrow t = \arcsin(x/a)$ $\quad -\pi/2 < t < \pi$