LECTURE 3 - TRIG INTEGRALS

Note Title

Start by recalling Pythagorean theorem in Trig form:

$$\sin^2\theta + \cos^2\theta = 1$$

Trig identities give another: $\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

These are the basic things to remember for trig integrals & trig sub.

<u>Trig</u> <u>Integrals</u>

Look at integrals of the form a) $\int \sin^m x \cos^n x \, dx$

and b) $\int \tan^m x \sec^n x \, dx$, m, n are integers.

a) basic idea: convert integral into one we can u-sub.

<u>Ex</u>: $\int \sin^5 x \, dx$ $\qquad \sin^5 x = (\sin x)\cdot(\sin^2 x)^2 = (\sin x)(1-\cos^2 x)^2$

$= \int \sin x \cdot (1-\cos^2 x)^2 \, dx$ $\qquad u = \cos x$, $du = -\sin x \, dx$

$= -\int (1-u^2)^2 \, du = -\int 1 - 2u^2 + u^4 \, du = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C$

$= \boxed{-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C}$

What did we do?

- exponent was odd: write as $(\sin x)(\sin^2 x)^{\text{something}}$
- $\sin^2 x = 1 - \cos^2 x$: $(\sin x)(1-\cos^2 x)^?$
- $u = \cos x \Rightarrow du = -\sin x \, dx \Rightarrow (1-u^2)^?$

<u>Ex</u>: $\int \cos^3 x \sin^2 x \, dx$ $\qquad \cos x \cdot \cos^2 x \cdot \sin^2 x = \cos x \cdot (1-\sin^2 x) \cdot \sin^2 x$

$= \int \cos x \, (1-\sin^2 x) \sin^2 x \, dx$ $\qquad u = \sin x \Rightarrow du = \cos x \, dx$

$$= \int (1-u^2)u^2\,du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$$

$$= \boxed{\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C}$$

So if either exponent is odd, same method.

What if neither is? Half angle formula:

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

Both are $\cos 2\theta$! "sine gets the sign"

Ex: $\displaystyle\int \cos^2\theta\,d\theta = \int \frac{1}{2} + \frac{1}{2}\cos 2\theta\,d\theta = \boxed{\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C}$

Ex: $\displaystyle\int \sin^2\theta\cos^2\theta\,d\theta$

$$\sin^2\theta\cos^2\theta = (1-\cos^2\theta)\cdot\cos^2\theta$$
$$= \cos^2\theta - \cos^4\theta$$
$$= \frac{1}{2} + \frac{1}{2}\cos 2\theta - \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)^2$$
$$= \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) - \left(\frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta\right)$$
$$= \frac{1}{4} - \frac{1}{8}(1+\cos(4\theta))$$
$$= \frac{1}{8} - \frac{1}{8}\cos(4\theta)$$

$$= \int \frac{1}{8} - \frac{1}{8}\cos 4\theta\,d\theta =$$

$$\boxed{\frac{1}{8}\theta - \frac{1}{32}\sin 4\theta + C}$$

Not immediately obvious that this is related!

<u>Method</u> Look at exponents:

1) If at least one is odd (say sin), then write integrand as $(\sin x)(1-\cos^2 x)^? \cos^n x$ (etc). Then $u = \cos x$.

2) If both even, then put everything in terms of one & use half-angle. Repeat. awful case!

Sec & Tan work largely the same way: Just a little trickier

$$\frac{d}{dx}\sec x = \sec x\tan x, \qquad \frac{d}{dx}\tan x = \sec^2 x$$

Ex $\displaystyle\int \sec^4 x\,dx$

$$\sec^4 x = \sec^2 x\cdot\sec^2 x$$
$$= (\tan^2 x + 1)\cdot\sec^2 x$$

$$= \underbrace{\int \tan^2 x \cdot \sec^2 x \, dx}_{\begin{array}{c} u=\tan x \\ du=\sec^2 x \end{array} \quad \int u^2\,du = \frac{1}{3}u^3} + \underbrace{\int \sec^2 x \, dx}_{\tan x} = \boxed{\frac{1}{3}\tan^3 x + \tan x + C}$$

This works for any integral of the form $\int \tan^m x \sec^{2k} x \; dx$:

write $\sec^{2k} x$ as $\sec^2 x \cdot (1+\tan^2 x)^{k-1}$ and $u = \tan x$.

Ex $\int \tan^3 x \sec x \; dx$

$$\tan^3 x \sec x = \underset{\underset{(\sec^2 x - 1)}{\parallel}}{\tan^2 x} \cdot \underbrace{\tan x \sec x}_{(\sec x)'}$$

$$= \int (\sec^2 x - 1) \cdot \tan x \sec x \; dx$$

$u = \sec x \Rightarrow du = \sec x \tan x \; dx$

$$= \int u^2 - 1 \; du = \frac{1}{3}u^3 - u + C$$

$$= \boxed{\frac{1}{3}\sec^3 x - \sec x + C}$$

This works for any integral of the form $\int \tan^{2k+1} x \; \sec^{n} x \; dx$: $n \geq 1$

$$\tan^{2k+1} x = \tan x \cdot (\sec^2 x - 1)^k \Rightarrow \int (\sec^2 x - 1)^k \cdot \sec^{n-1} x \cdot \sec x \tan x \, dx.$$