Lecture 27 - Convergence

Note Title

Had defined $T_m(x) = \sum_{n=0}^{m} \frac{f^{(n)}(b)}{n!}(x-b)^n$ & $R_m(x) = f(x) - T_m(x)$.

Also said that $f(x)$ equals its Taylor series at x if $\lim_{m\to\infty} R_m(x) = 0$, because $\lim_{m\to\infty} T_m(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!}(x-b)^n$ by def.

<u>Thm</u> Let M be such that for all $|x-b|<d$

$$|f^{(m+1)}(x)| \le M. \text{ Then}$$

$$|R_m(x)| \le \frac{M\,|(x-b)|^{m+1}}{(m+1)!} \leftarrow \text{same form as } (m+1)^{st} \text{ term!}$$

<u>Ex</u>: $f(x) = e^x$, $b=0$. Then for all $|x| \le d$,

$$|e^x| \le e^d.$$

So $|R_m(x)| \le \frac{e^d\,|x|^{m+1}}{(m+1)!}$.

Now $\lim_{m\to\infty} \frac{|x|^{m+1}}{(m+1)!} = 0$ for all x.

(2 reasons: ① eventually, $m > |x|$, so $1 > \frac{|x|}{m} > \frac{|x|}{m+1} > \dots$ & product $\to 0$

② $\sum \frac{|x|^{m+1}}{(m+1)!}$ converges $\Rightarrow \frac{|x|^{m+1}}{(m+1)!} \to 0$.)

So by squeeze theorem, $R_m(x) \to 0$ for all x & letting d vary

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x.$$

<u>Ex</u> $f(x) = \sin x$. $b=0$

Then $|f^{(m+1)}(x)| = \begin{cases} |\sin x| & m \text{ odd} \\ |\cos x| & m \text{ even} \end{cases} \le 1$

So $|R_m(x)| \le \frac{1\,|(x)|^{m+1}}{(m+1)!} = \frac{|x|^{m+1}}{(m+1)!} \to 0$ for all x

$$\Rightarrow \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for all } x.$$

Get for free that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

by applying $\frac{d}{dx}(-)$ to both side of $\sin(x)$ story.

Can use this formula to approximate $f(x)$ to within any error

Ex: Approx e^1 to within $.1$

$e^1 \leq 3$, so on $|x| \leq 1$, $|f^{(m+1)}(x)| \leq 3$ (can use anything bigger than $f^{(m+1)}(x)$. 3 is easier). So $|R_m(1)| \leq \frac{3 \cdot 1^{m+1}}{(m+1)!} = \frac{3}{(m+1)!}$

m	$R_m(1)$	$\leq .1$?	$T_m(1)$	
0	3	No	1	(1)
1	$3/2$	No	$1+1$	$(1+x)$
2	$3/3! = 1/2$	No	$1+1+\frac{1}{2}$	$(1+x+\frac{x^2}{2})$
3	$3/4! = 1/8$	No	$1+1+\frac{1}{2}+\frac{1}{6}$	$(1+x+\frac{x^2}{2}+\frac{x^3}{6})$
4	$3/5! = 1/40$	Yes!	$1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}$	$($ " $+\frac{x^4}{4!})$.

e^1 to within $1/40$!

Ex sin(.1) to within .001: Again: $M = 1$ works

$$R_m(.1) = \frac{1 \cdot (.1)^{m+1}}{(m+1)!}$$

m	$R_m(.1)$	$\leq .001$?	$T_m(.1)$	
0	$.1$	No	0	(0)
1	$1/200$	No	$.1$ same!	$(0+x)$
2	$1/6{,}000$	Yes	$.1$	$(0+x+0)$
3	$1/240{,}000$	Yes	$.1 - \frac{1}{6{,}000}$	$(0+x+0-\frac{x^3}{3!})$

See that $R_m(x)$ can be a very coarse approx. We are making quite conservative estimates.

Integrals

Can use power series to find integrals.

If $f(x) = \sum a_n x^n$, then $\int f(x)\,dx = \sum \frac{a_n}{n+1} x^{n+1}$.

Ex: $f(x) = e^{x^3}$. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow$

$$f(x) = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

Then $\int e^{x^3}\,dx = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)\cdot n!}$.

We have a power series, and this is the best we can do.

Ex $\sin(x^2)$: $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\Rightarrow \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

So $\int \sin(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$.