Lecture 25 - Taylor Series

Note Title

Gives us a method to write any function as a Taylor series.

Thm If $f(x) = \sum_{n=0}^{\infty} a_n (x-b)^n$ has radius of convergence $R>0$, then

$$a_n = \frac{f^{(n)}(b)}{n!}.$$

Q: Where does the $n!$ come from? Why $f^{(n)}(b)$?

A: Take m derivatives of both sides:

$$\frac{d^m}{(dx)^m} f(x) = \frac{d^m}{(dx)^m}\left(a_0 + a_1(x-b) + a_2(x-b)^2 + \dots\right) \qquad (*)$$

Prop a) If $k<m$, then $\left(\frac{d}{dx}\right)^m x^k = 0$ — implies by taking one more $\frac{d}{dx}$.

b) If $k=m$, then $\left(\frac{d}{dx}\right)^m x^m = m!$ — special case

c) If $k>m$, then $\left(\frac{d}{dx}\right)^m x^k = \frac{k!}{(k-m)!} x^{k-m}$ ← easy induction

Ex: $x^4 \longmapsto 4x^3 \longmapsto 4\cdot 3x^2 \longmapsto 4\cdot3\cdot2x \longmapsto 4\cdot3\cdot2\cdot1 \longmapsto 0$, with $x^4 \xrightarrow{\left(\frac{d}{dx}\right)^4} 4\cdot3\cdot2\cdot1$

So $\left(\frac{d}{dx}\right)^m f(x) = m!\,a_m + \frac{(m+1)!}{1!} a_{m+1}(x-b) + \frac{(m+2)!}{2!} a_{m+2}(x-b)^2 + \dots$

$\leadsto$ @ $x=b$: $m!\,a_m + \frac{(m+1)!}{1!} a_{m+1}\cancelto{0}{(b-b)} + \frac{(m+2)!}{2!} a_{m+2}\cancelto{0}{(b-b)^2 + \dots}$

$\Rightarrow$ $f^{(m)}(b) = m!\,a_m$

$\leftrightarrow$ $a_m = \frac{f^{(m)}(b)}{m!}.$

Having a non-zero Radius of Convergence $\Rightarrow \sum_{n=0}^{\infty} a_n(x-b)^n$ is diff'able & derivative has same R of C → can take $\frac{d}{dx}$ as many times as we want. ⇒ all makes sense.

Def The Taylor series of a function $f(x)$ centered at b is the power series $\sum_{n=0}^{\infty} a_n (x-b)^n$ where $a_m = \frac{f^{(m)}(b)}{m!}$ for all m.

If $b=0$, say it is the Maclaurin Series.

Ex: $f(x)=e^x$, $b=0$:

Must find $a_m = f^{(m)}(0)/m!$:

$\frac{d}{dx}e^x = e^x \Rightarrow f^{(m)}(x) = e^x$ for all $m \Rightarrow f^{(m)}(0)=1$

$\Rightarrow a_m = 1/m!$. $\Rightarrow$ $$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

Table:

m	$f^{(m)}(x)$	$f^{(m)}(b)$	a_m
0	e^x	1	1
1	e^x	1	1
2	e^x	1	$1/2$
3	e^x	1	$1/3!$
4	e^x	1	$1/4!$
	$\vdots$	$\vdots$	$\vdots$

What is the radius of converce?

Look at $$\frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \left|\frac{x^{\cancel{n+1}}}{\cancel{x^n}}\right| \cdot \frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}} = \frac{|x|}{n+1}$$

As $n \to \infty$, $\frac{|x|}{n+1} \to 0$ which is always <1

$\Rightarrow$ absolute convergence for all x. R of C is ∞.

Ex: $f(x) = \sin x$, $b=2\pi$

Table:

m	$f^{(m)}(x)$	$f^{(m)}(b)$	a_m
0	$\sin x$	0	0
1	$\cos x$	1	1
2	$-\sin x$	0	0
3	$-\cos x$	-1	$-1/3!$
start over $\rightarrow$ 4	$\sin x$	0	0

$$\Rightarrow \sin x = (x-2\pi) - \frac{1}{3!}(x-2\pi)^3 + \frac{1}{5!}(x-2\pi)^5 - \frac{1}{7!}(x-2\pi)^7 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-2\pi)^{2n+1}}{(2n+1)!}$$

@ $b=0$: $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

For the R of C:

$$\frac{\left|\frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!}\right|}{\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right|} = \left|\frac{x^{\cancel{2n+3}\,2}}{\cancel{x^{2n+1}}}\right| \frac{\cancel{(2n+1)!}}{(2n+3)\cdot(2n+2)\cdot\cancel{(2n+1)!}}$$

$$= \frac{x^2}{(2n+3)(2n+2)} \xrightarrow[n\to\infty]{} 0 < 1 \quad \text{for all } x.$$

So ratio test $\Rightarrow$ the R of C is ∞.

Ex $f(x) = \cos x \qquad b=0$

m	$f^{(m)}(x)$	$f^{(m)}(b)$	a_m
0	$\cos x$	1	1
1	$-\sin x$	0	0
2	$-\cos x$	-1	$-1/2!$
3	$\sin x$	0	0
repeats $\rightarrow$ 4	$\cos x$	1	$1/4!$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$ also has ∞ R of C.

Def: •) The m^{th} Taylor polynomial centered at b of $f(x)$ is

$$T_m(x) = \sum_{n=0}^{m} \frac{f^{(n)}(b)}{n!}(x-b)^n.$$

•) The m^{th} remainder is

$$R_m(x) = f(x) - T_m(x).$$

Thm A function equals its Taylor series at x if $\lim_{m\to\infty} R_m(x) = 0$.