Lecture 24 — Functions as Power Series

Already saw that \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \).

Thm If \( f(x) = \sum_{n=0}^{\infty} a_n (x-b)^n \) has a non-zero radius of convergence \( R \), then

1. \( f(x) \) is differentiable \( \uparrow \)

\[ f'(x) = \sum_{n=0}^{\infty} a_n \cdot n \cdot (x-b)^{n-1}, \text{ with radius of convergence } R \]

2. \( f(x) \) is integrable \( \uparrow \)

\[ \int f(x) \, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \text{ with radius of convergence } R. \]

So we can use our known power series:

\[ \frac{1}{1-(\frac{1}{a})} = \sum_{n=0}^{\infty} \left( \frac{x}{a} \right)^n = \sum_{n=0}^{\infty} \frac{x^n}{a^n} \]

The radius of convergence of \( \frac{1}{1-x} \) is 1, so the radius of convergence of \( \frac{1}{1-(\frac{1}{a})} \) is \( a \).

Ex: \( \frac{1}{(1-x)^2} = \frac{d}{dx} \left( \frac{1}{1-x} \right) = \sum_{n=0}^{\infty} n \cdot x^{n-1} = 0 + 1 + 2x + 3x^2 +... \)

(reindexed)

\[ \sum_{n=0}^{\infty} (n+1) x^n \]

\[ \frac{2}{(1-x)^3} = \frac{d}{dx} \left( \frac{1}{(1-x)^2} \right) = \sum_{n=0}^{\infty} (n+1) \cdot n \cdot x^{n-1} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)x^n}{(n+2)(n+1)} \]

The radius of convergence in all cases is still 1.

Quick Review: How do we find this? Ratio test.

\( \sum_{n=0}^{\infty} (n+1)x^n \): look at \( \lim_{n \to \infty} \left| \frac{(n+2)x^{n+1}}{(n+1)x^n} \right| = \lim_{n \to \infty} \left| \frac{n+2}{n+1} \right| x < 1 \)

\( \longleftrightarrow |x| < 1 \leftrightarrow R_{\text{of C}} = 1 \).

Can get more complicated functions:

\[ \frac{x}{(1-x)^2} = x \cdot \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n \cdot x^n, \text{ etc} \]
Also get integrals:

\[ \int \frac{1}{1 + x} \, dx = \frac{1}{1 - (-x)} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{Ref: C = 1} \]

So \( \ln(1 + x) = \int \frac{1}{1 + x} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \quad \text{Ref: C = 1} \)

This series converges absolutely for \( |x| < 1 \).

Have to check what happens at \( |x| = 1 \):

\( x = 1 \): \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \), Series is alternating, \( b_n = \frac{(-1)^n}{n+1} \) is decreasing \( \implies \lim_{n \to \infty} b_n = 0 \).

\( \Rightarrow \) converges

\( x = -1 \): \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \cdot (-1)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1} \) diverges.

\( \therefore x = 1 \) learn that \( \ln(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \).

So while \( \frac{1}{1 + x} \) converges only on \( (-1, 1) \),

\( \ln(1 + x) \) converges on \( (-1, 1) \).

\[ \int \frac{1}{1 + x^2} \, dx = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \]

\( \tan^{-1}(x) = \int \frac{1}{1 + x^2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \)

\( \frac{1}{1 + x^2} \) converges for \( |x^2| < 1 \) \( \iff \) \( -1 < x < 1 \)

So \( \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \) converges for \( -1 < x < 1 \)

\( \therefore x = 1 \): \( \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \) converges by alternating series test.

\( \therefore x = -1 \): \( \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot (-1)^{2n+1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \) converges by alternating series test.
Learn that \( \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \) converges on \([-1, 1]\).

If \( x = 1 \), learn that
\[
\frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}. 
\]

Remarks on radius of convergence:

1. Important that we have absolute convergence. This essentially lets us commute limits if claim continuity.

2. Absolute convergence is an "open" condition: if it converges absolutely at a point, it does so near the point, hence having on open interval's worth of points.

3. Usually easy to find the ROC using ratio test.