Lecture 24 — Functions as Power Series

Note Title

Already saw that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

<u>Thm</u> If $f(x) = \sum_{n=0}^{\infty} a_n (x-b)^n$ has a non-zero radius of convergence R, then

① $f(x)$ is differentiable &

$$f'(x) = \sum_{n=0}^{\infty} a_n \cdot n\, x^{n-1}, \quad \text{with radius of convergence } R$$

② $f(x)$ is integrable &

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \quad \text{with radius of convergence } R.$$

So we can use our known power series: $\frac{1}{1-(x/a)} = \sum_{n=0}^{\infty} \left(\frac{x}{a}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{a^n}$

The radius of convergence of $\frac{1}{1-x}$ is 1, so the radius of convergence of $\frac{1}{1-(x/a)}$ is a.

<u>Ex</u>: $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=0}^{\infty} n x^{n-1} = 0 + 1 + 2x + 3x^2 + \dots$

(reindexed) $= \sum_{n=0}^{\infty} (n+1) x^n$

$$\frac{2}{(1-x)^3} = \frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \sum_{n=0}^{\infty} (n+1)\cdot n\, x^{n-1} = \sum_{n=0}^{\infty} (n+2)(n+1) x^n$$

The radius of convergence in all cases is still 1.

<u>Quick</u> <u>Review</u> How do we find this? Ratio test.

$\sum_{n=0}^{\infty} (n+1)x^n$: look at $\lim_{n\to\infty} \left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = \lim_{n\to\infty} \left|\frac{n+2}{n+1}\right| x < 1$

$\longleftrightarrow |x| < 1 \leftrightarrow$ R of C = 1.

Can get more complicated functions: $\frac{x}{(1-x)^2} = x \cdot \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^n$, etc

Also get integrals:

<u>Ex</u>: $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n$ R of C = 1

So $\ln(1+x) = \int \frac{1}{1+x}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$ R of C = 1

This series converges absolutely for $|x|<1$.

Have to check what happens at $|x|=1$:

$x=1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$. Series is alternating. $b_n = \left|\frac{(-1)^n}{n+1}\right| = \frac{1}{n+1}$ is decreasing & $\lim_{n\to\infty} b_n = 0$.

$\Rightarrow$ converges

$x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(-1)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

@ $x=1$: learn that $\ln(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$.

So while $\frac{1}{1+x}$ converges only on $(-1,1)$,

$\ln(1+x)$ converges on $(-1,1]$.

<u>Ex</u>: $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$\tan^{-1}(x) = \int \frac{1}{1+x^2}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

$\frac{1}{1+x^2}$ converges for $|-x^2|<1 \longleftrightarrow -1<x<1$

So $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ converges for $-1<x<1$

@ $x=1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges by alternating series test.

@ $x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(-1)^{2n+1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges by alternating series test.

Learn that $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ converges on $[-1, 1]$.

If $x = 1$, learn that

$$\frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Remarks on radius of convergence:

① Important that we have absolute convergnce. This essentially lets us commute limits & claim continuity.

② Absolute convergence is an "open" condition: if it converges absolutely at a point, it does so near the point. hence having an open interval's worth of points.

③ Usually easy to find the R of C using ratio test.