# Lecture 23 - Power Series

Note Title

Going to start approximating functions by power series.

Basic Example:

If $|r|<1$, then $1+r+r^2+\ldots = \frac{1}{1-r}$ $\leftrightarrow$ If $|x|<1 \Rightarrow 1+x+\ldots = \frac{1}{1-x}$

This holds for all such $x$ (and continuously so), so we have found a way to express $f(x) = \frac{1}{1-x}$ as an infinite series in $x$.

<u>Def</u> A <u>power series</u> is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-b)^n$$

Say that $b$ is the "center" of the power series.

Goal for the remainder of the course is to find power series expansions of functions.

<u>Ex</u>: Have already $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Can use this to get other series:

$$\frac{1}{1-x^2} = 1 + (x^2) + (x^2)^2 + \ldots = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + \cdots = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\frac{1}{2-x} = \frac{1}{1-(x-1)} = 1 + (x-1) + (x-1)^2 + \ldots = \sum_{n=0}^{\infty} (x-1)^n \qquad \text{(centred at 1)}$$

<u>Ex</u>: Find a power series representing $\frac{x}{1-x^2}$.

$$\frac{x}{1-x^2} = x + x\cdot x^2 + x\cdot(x^2)^2 + \ldots = \boxed{\sum_{n=0}^{\infty} x^{2n+1}}$$

$a_0$

$r$

Why power series? Two big uses:

① Approximating values of tough functions
(these are how computers compute things like $e^x$, $\sin x$, etc)
② Integrate functions we couldn't do before ($\int e^{x^2} dx$, etc)
$\rightsquigarrow$ can get "explicit" solutions to differential equations.

Want to know when our series give meaningful results $\longleftrightarrow$ when they converge.

Thm One of three things happens:

① $\sum_{n=0}^{\infty} a_n (x-b)^n$ converges only at $x=b$

② There is an R s.t. $\sum_{n=0}^{\infty} a_n (x-b)^n$ converges absolutely for $|x-b|<R$

③ $\sum_{n=0}^{\infty} a_n (x-b)^n$ converges absolutely for all x.

In case 2, we also must consider whether the series converges for $|x-b|=R$.

Def R in the above theorem is the radius of convergence.

Ex Already saw that $1+x+\dots$ converges if and only if $|x|<1$

So for any series like this, the radius of convergence is 1:

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n \quad \text{converges for} \quad |x|<1$$

$$\frac{1}{2-x} = \sum_{n=0}^{\infty} (x-1)^n \quad \text{converges for} \quad |(x-1)|<1 \quad \longleftrightarrow \quad 0<x<2$$

We normally find the radius of convergence with the ratio test.

Quick reminder: If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} <1$, then the series converges absolutely

$>1$, then the series diverges

Apply this to $a_n(x-b)^n$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-b)^{\cancel{n+1}}}{a_n\cancel{(x-b)^n}}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\cdot|x-b|$$

Independent of n!

3 cases

(a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$. Then $\lim_{n\to\infty}\left|\frac{a_{n+1}(x-b)}{a_n}\right| < 1$ only when

$x-b=0 \Rightarrow \sum_{n=0}^{\infty} a_n(x-b)^n$ converges only at $x=b$

(b) $0 < \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r < \infty$. Then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x-b| < 1$ if and only if

$|x-b| < \frac{1}{r} = R \Rightarrow \sum_{n=0}^{\infty} a_n(x-b)^n$ converges if $|x-b| < R$.

(c) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 \Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x-b| = 0$ for all $x$

$\Rightarrow \sum_{n=0}^{\infty} a_n(x-b)^n$ converges for all $x$.

Ex: $\sum_{n=0}^{\infty} n x^n$ :

$a_n = n$, so $\frac{a_{n+1}x^{n+1}}{a_n x^n} = \frac{n+1}{n}x \Rightarrow \lim_{n\to\infty}\left|\frac{n+1}{n}x\right| = |x|$

This is $<1 \Leftrightarrow |x|<1$. So radius of convergence is 1.

$\sum_{n=0}^{\infty} n!\, x^n$ : $a_n = n!$ so $\frac{a_{n+1}x^{n+1}}{a_n x^n} = \frac{(n+1)!\, x^{n+1}}{n!\; x^n} = (n+1)x$

If $x \neq 0$, then $\lim_{n\to\infty}|(n+1)x| = \infty$, so converges only at $x=0$.

Ex Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{n}{3^n}(x-2)^n$.

$$\left|\frac{a_{n+1}(x-2)^{n+1}}{a_n(x-2)^n}\right| = \left|\frac{\left(n+1/3^{n+1}\right)}{\left(n/3^n\right)}(x-2)\right| = \left|\left(\frac{n+1}{n}\right)\cdot\left(\frac{x-2}{3}\right)\right| \longrightarrow \frac{|x-2|}{3}$$

So $\lim_{n\to\infty}\left|\frac{a_{n+1}(x-2)^{n+1}}{a_n(x-2)^n}\right| < 1 \longleftrightarrow |x-2| < 3$. Rad of conv. $= 3$.

What about at end points? Plug-in and apply any test

Ex: $\sum_{n=1}^{\infty} \frac{x^n}{n}$ has a radius of convergence of 1. Have to check $|x|=1$

$x=1$: $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, $x=-1$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges.