Lecture 22 - Absolute Convergence, Ratio & Root Tests

Note Title

Def: A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

If $\sum a_n$ is convergent but $\sum |a_n|$ diverges, the series is conditionally convergent.

Ex: •) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

•) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Thm If $\sum a_n$ is absolutely convergent, then it converges.

Pf: $0 \le a_n + |a_n| \le 2|a_n|$

$b_n = a_n + |a_n|$ is non-negative and $\le 2|a_n|$ for all n. Since $\sum 2|a_n|$ converges, we get

$$\sum b_n = \sum (a_n + |a_n|) = \sum a_n + \sum |a_n| \quad \text{converges}$$

$\Rightarrow \sum a_n$ converges.

Have 2 Huge Tests for convergence:

1. Ratio Test
2. Root Test

Thm Let $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$.

•) If $L < 1$, then the series $\sum a_n$ is absolutely convergent

•) If $L > 1$, then the series $\sum |a_n|$ diverges

•) If $L = 1$, then the test is inconclusive.

Idea: If $\frac{|a_{n+1}|}{|a_n|} \to L$ then for all n sufficiently large, we can compare to a geometric series w/ $r < 1$.

Comparison test $\Rightarrow$ convergence

$\Rightarrow \sum |a_n|$ converges by comparison.

This is most useful with "factorial"

Def If $n \geq 1$, then $n! = n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$

If $n=0$, then $n! = 1$.

Ex: $2! = 2$

$3! = 3\cdot 2\cdot 1 = 6$

$4! = 4\cdot 3\cdot 2\cdot 1 = 24$

$\vdots$

$n! = n\cdot(n-1)!$

Last part shows why the root test is easy to apply here.

Ex: Let a_n be the sequence $a_n = \frac{1}{n!} \rightsquigarrow \sum a_n$ is absolutely convergent

Then $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim\limits_{n\to\infty} \frac{\frac{1}{(n+1)!}}{1/n!} = \lim\limits_{n\to\infty} \frac{n!}{(n+1)!} = \lim\limits_{n\to\infty} \frac{1}{n+1} = 0 < 1$

$\Rightarrow$ absolutely convergent.

Ex: Determine if $\sum \frac{3^n}{n!}$ is convergent.

Ratio Test: $\frac{|a_{n+1}|}{|a_n|} = \frac{3^{n+1}/(n+1)!}{3^n/n!} = \frac{3}{n+1} \longrightarrow 0$ as $n\to\infty$

This is $< 1 \Rightarrow \sum 3^n/n!$ is absolutely convergent.

One final test for convergence: root test. Used least often

Thm Let $L = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$. Then If

$L<1$, $\sum a_n$ converges absolutely

$L>1$, $\sum |a_n|$ diverges

$L=1$, test is inconclusive.

Ex: $a_n = \frac{1}{n^n}$ Then $\sum a_n$ converges absolutely

Root test: $\sqrt[n]{|a_n|} = \left(\frac{1}{n^n}\right)^{1/n} = \frac{1}{n} \longrightarrow 0$ as $n\to\infty$

Ex: $a_n = \frac{1}{2^n+1}$.

Root test: $\sqrt[n]{|a_n|} = \left(\frac{1}{2^n+1}\right)^{1/n} \longrightarrow L$

Of the form 0^∞, so use ln then L'Hôpital's rule:

$$\frac{-\ln(2^n+1)}{n} \longrightarrow \ln L$$

$$\rightsquigarrow \frac{\frac{-2^n \ln 2}{2^n+1}}{1} \longrightarrow -\ln 2 \Rightarrow L = 1/2 < 1.$$

Seen lots of tests. How do we pick?

The order we normally check:

① If $\lim_{n\to\infty} a_n \neq 0$, diverges

② If a_n swaps signs: alternating series test $|a_n|$ decreasing & $a_n \to 0$

If a_n's sign changes weirdly, test $|a_n|$.

For positive a_n (or for $|a_n|$):

③ Ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow$ convergence, $> 1 \Rightarrow$ divergence.

If we see factorials, we almost always use this.

④ Comparison tests $a_n \le b_n$ & $\sum b_n$ converges $\Rightarrow \sum a_n$ converges &

$0 < \lim b_n/a_n < \infty \longleftrightarrow$ both converge or diverge

If our sequence look like a geometric or p-series, use this.

Use the final two less often. They are usually harder to use.

⑤ If a_n look like $f(n)$ some decreasing f we can integrate

$\Rightarrow$ Integral test $\sum a_n$ converges $\longleftrightarrow \int_k^\infty f(x)\,dx$ converges

⑥ Root test $\lim \sqrt[n]{|a_n|} < 1 \Rightarrow$ converges, $> 1 \Rightarrow$ diverges

If we see n^n (and we can't easily compare), use this

Ex: What test to try:

ⓐ $\frac{|\sin n|}{2^n}$: comparison w/ $\frac{1}{2^n}$

ⓒ $\frac{(-1)^n}{2n-1}$ alternating series

ⓑ $\frac{1}{n^2+1}$: compare w/ $1/n^2$ (or $\int$)

ⓓ $\frac{2^{2n+1}}{n!}$ ratio test