Def: A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

If $\sum a_n$ is convergent but $\sum |a_n|$ diverges, the series is conditionally convergent.

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Thm: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it converges.

Pf: $0 \leq |a_n + |a_n|| \leq 2|a_n|

$b_n = a_n + |a_n|$ is non-negative and $\leq 2|a_n|$ for all $n$. Since $\sum 2|a_n|$ converges, we get

$\sum b_n = \sum (a_n + |a_n|) = \sum a_n + \sum |a_n|$ converges

$\Rightarrow \sum a_n$ converges.

Have 2 Huge Tests for convergence:

1) Ratio Test
2) Root Test

Thm: Let $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L$.

1) IF $L<1$, then the series $\sum |a_n|$ is absolutely convergent.
2) IF $L>1$, then the series $\sum |a_n|$ diverges.
3) IF $L=1$, then the test is inconclusive.

Idea: IF $\frac{|a_{n+1}|}{|a_n|} \to L$ for all $n$ sufficiently large,
we can compare to a geometric series w/ $r < 1$.

Comparison test $\Rightarrow$ convergence.
\[ \sum |a_n| \text{ converges by comparison.} \]

This is most useful with "factorial"

**Def** If \[ n \geq 1, \text{ then } n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 \]

If \[ n = 0, \text{ then } n! = 1. \]

**Ex:**
\[ 2! = 2 \]
\[ 3! = 3 \cdot 2 \cdot 1 = 6 \]
\[ 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \]
\[ \vdots \]
\[ n! = n \cdot (n-1)! \]

Last part shows why the root test is easy to apply here.

**Ex:** Let \[ a_n = \frac{1}{n!} \rightarrow \sum a_n \text{ is absolutely convergent} \]

Then \[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{1}{(n+1)!} = \lim_{n \to \infty} \frac{n+1}{n!} = \lim_{n \to \infty} \frac{1}{n+1} = 0 < 1
\]

\[ \Rightarrow \text{ absolutely convergent.} \]

**Ex:** Determine if \[ \sum \frac{3^n}{n!} \text{ is convergent.} \]

Ratio Test: \[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{3^{n+1}/(n+1)!}{3^n/n!} = \frac{3}{n+1} \rightarrow 0 \text{ as } n \to \infty \]

This is \( < 1 \Rightarrow \sum \frac{3^n}{n!} \text{ is absolutely convergent.} \)

One final test for convergence: root test. Used least often

**Thm** Let \[ L = \lim_{n \to \infty} \sqrt[n]{|a_n|}. \text{ Then if} \]
\[ L < 1, \text{ } \sum a_n \text{ converges absolutely} \]
\[ L > 1, \text{ } \sum |a_n| \text{ diverges} \]
\[ L = 1, \text{ test is inconclusive.} \]

**Ex:** \[ a_n = \frac{1}{n^n} \text{ Then } \sum a_n \text{ converges absolutely} \]

Root test: \[ \sqrt[n]{|a_n|} = \left( \frac{1}{n^n} \right)^{1/n} = \frac{1}{n} \rightarrow 0 \text{ as } n \to \infty \]

**Ex:** \[ a_n = \frac{1}{n^{n!}} \]

Root test: \[ \sqrt[n]{|a_n|} = \left( \frac{1}{n^{n!}} \right)^{1/n} \rightarrow 1 \]
Of the form $\frac{c^n}{n}$, so use $\ln \ln$, L'Hopital's rule:

$$-\frac{\ln(2^{n+1})}{n} \rightarrow \ln 1$$

$$\sim -\frac{2^n \ln 2}{2^n+1} \rightarrow -\ln 2 \Rightarrow L = \frac{1}{2} < 1.$$ 

Seen lots of tests. How do we pick?

The order we normally check:

1. If $\lim_{n\to\infty} a_n \neq 0$, diverges
2. If $a_n$ swaps signs: alternating series test (or decreasing) if $a_n \to 0$.
   If $a_n$'s sign changes weirdly, test $|a_n|$.

   For positive $a_n$ (or for $|a_n|$):

3. Ratio test $\lim_{n\to\infty} \frac{|a_{n+1}|}{a_n} < 1 \Rightarrow$ converges, $>1 \Rightarrow$ diverges.

   If we see factorials, we almost always use this.

4. Comparison tests $a_n \leq b_n \Rightarrow \sum b_n$ converges $\Rightarrow \sum a_n$ converges

   $0 < \lim_{n\to\infty} \frac{b_n}{a_n} < \infty \Leftrightarrow$ both converge or diverge

   If our sequence look like a geometric or p-series, use this.

   Use the final two less often. They are usually harder to use.

5. If $a_n$ look like $f(n)$ some decreasing $f$ we can integrate

   $\Rightarrow$ integral test $\sum a_n$ converges $\Leftrightarrow \int_1^\infty f(x) \, dx$ converges

6. Root test $\lim \sqrt[n]{|a_n|} < 1 \Rightarrow$ converges, $>1 \Rightarrow$ diverges

   If we see $n^a$ (and we can't easily compare), use this

Ex: What test to try:

1. $\frac{|\sin n|}{2^n}$: comparison w/ $\frac{1}{2^n}$
2. $\frac{(-1)^n}{2^{n-1}}$: alternating series

3. $\frac{1}{n^2+1}$: compare w/ $\frac{1}{n^2}$ (or $\int$)
4. $\frac{2^n+1}{n!}$: ratio test