Lecture 21 - Comparison Test & Alternating Series

Note Title

Comparing two series is one of the most useful techniques.

Moral: If it essentially looks like something that converges, it converges.

Thm Let a_n and b_n be positive sequences. Then

1) If $a_n \le b_n$ for n sufficiently large & $\sum b_n$ converges, then $\sum a_n$ converges.

2) If $a_n \ge b_n$ for n sufficiently large & $\sum b_n$ diverges, then $\sum a_n$ diverges.

Usually compare to a p series $\left(\sum \frac{1}{n^p}\right)$, a geometric series or one we can apply the integral test to.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}$ converges: $n(n+1)^2 \ge n^3$ for all $n \ge 1$, so $\frac{1}{n(n+1)^2} \le \frac{1}{n^3}$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ converges} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} \text{ converges}$$

Ex: $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-n}}$ $\qquad \sqrt{n^2-n} \le n$ for all $n \ge 1$, so $\frac{1}{\sqrt{n^2-n}} \ge \frac{1}{n}$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-n}} \text{ diverges.}$$

Why does this work?

① Since a_n is ≥ 0, the sequence $s_n = a_1 + \dots + a_n$ is increasing:

$$s_n = s_{n-1} + a_n \ge s_{n-1}.$$

② $a_n \le b_n \Rightarrow s_n = a_1 + \dots + a_n \le b_1 + \dots + b_n = t_n$ for all n.

The sequence t_n is an increasing sequence w/ limit $\sum b_n$, so

$s_n \le t_n \le \sum b_n$ for all n. $\Rightarrow$ s_n is bounded above.

So s_n is bounded above & increasing $\Rightarrow$ converges.

Divergence is easier. If $b_n \ge 0$ & $\sum b_n$ diverges then $\sum b_n$ is infinite.

$a_n \ge b_n \Rightarrow s_n \ge t_n$ & $t_n \to \infty \Rightarrow s_n \to \infty$.

There is a refined comparison test. The "limit comparison"

<u>Thm</u> Let a_n & b_n be positive sequences.
If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is non-zero, then either $\sum a_n$ and $\sum b_n$ both converge or both diverge.

The condition "$\lim_{n\to\infty} \frac{a_n}{b_n}$ exists & is non-zero" is the statement that a_n and b_n are essentially the same.

<u>Ex</u>: $\sum_{n=1}^{\infty} \frac{1}{n\ln(n+4)}$ diverges. $a_n = \frac{1}{n\ln(n+4)}$ looks like $b_n = \frac{1}{n\ln(n)}$.

& $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{n}\ln(n)}{\cancel{n}\ln(n+4)} = 1$. Integral test shows that $\sum b_n$ diverges (check this for yourself!) so $\sum \frac{1}{n\ln(n+4)}$ diverges.

<u>Ex</u>: $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$ converges: $a_n = \frac{1}{(2n+1)^2}$ looks like $b_n = \frac{1}{n^2}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{(2n+1)^2} = \lim_{n\to\infty} \frac{n^2}{4n^2+4n+1} = \frac{1}{4}$. Since $\sum \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$ converges.

Why does this work?

If the limit exists and is non-zero, then we can find $0 < m \le M$ s.t. $m \le \frac{a_n}{b_n} \le M$ for all n suff. big.

$\Rightarrow$ $m \cdot b_n \le a_n \le M \cdot b_n$.

$\sum a_n$ converges $\Downarrow$ $\sum b_n$ converges

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<u>Ex</u>: $\sum_{n=1}^{\infty} \frac{n^2}{n^3+3n+1}$ diverges: $a_n = \frac{n^2}{n^3+3n+1}$ looks like $b_n = \frac{1}{n}$

& $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{n^3+3n+1}$ diverges.

Alternating Series:

Def: A sequence is alternating if the signs of a_n and a_{n+1} are different for all n.

Ex: •) $a_n = (-1)^n e^{-n}$ •) $1, -2, 4, -8, 16, \ldots$

Thm If a_n is an alternating sequence, then

$\sum a_n$ converges if

① The sequence $b_n = |a_n|$ is decreasing

② $\lim_{n\to\infty} a_n = 0$

This is very easy to apply!

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. Series is alternating. $b_n = |a_n| = \frac{1}{n}$ is decreasing & $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges. Alternating Series Test:

① $b_n = \frac{1}{\ln(n)}$ is decreasing &

② $\lim_{n\to\infty} \frac{(-1)^n}{\ln(n)} = 0$.

Def A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges. If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ converges conditionally.

So $\sum \frac{(-1)^n}{n}$ converges only conditionally.

Why might we care?

If $\sum a_n$ converges absolutely, then any rearrangement of terms gives the same sum.

If $\sum a_n$ converges conditionally, then we can rearrange the terms of the series to get any sum we like!