Lecture 20 - Integral Test

Note Title

Begin now a series of tests to help us determine if a series converges or diverges. This will <u>not</u> tell us the sum! First, an observation.

A series $\sum_{n=k}^{\infty} a_n$ converges if and only if $\sum_{n=m}^{\infty} a_n$ converges for some $m \geq k$.

In other words convergence cares only about the eventual behavior of the sequence. Argument is simple:

$$\sum_{n=k}^{\infty} a_n = \underbrace{\left(a_k + a_{k+1} + \cdots + a_{m-1}\right)}_{\text{just a number}} + \sum_{n=m}^{\infty} a_n$$

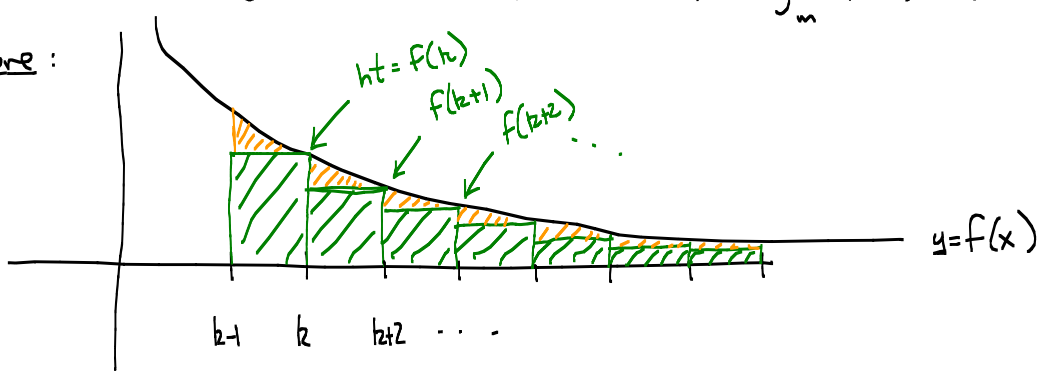
$\Rightarrow$ no influence on convergence.

So it always suffices to show $\sum_{n=m}^{\infty} a_n$ converges for m big.

<u>Integral Test</u>:

<u>Idea</u>: Our series is a Riemann sum for $\int_m^{\infty} f(x)\,dx$.

<u>Picture</u>:



Green Area = $\sum_{n=k}^{\infty} f(n)$

Green + Orange Area = $\int_{k-1}^{\infty} f(x)\,dx \geq \sum_{n=k}^{\infty} f(n)$

<u>Integral Test</u>: If there is a positive, decreasing function $f(x)$ s.t. $a_n = f(n)$, then $\sum a_n$ converges if and only if $\int_k^{\infty} f(x)\,dx$ converges for some k.

Picture shows the convergence part: If $\int_k^\infty f(x)\,dx$ converges then the orange & green area is finite, so the green area is finite.

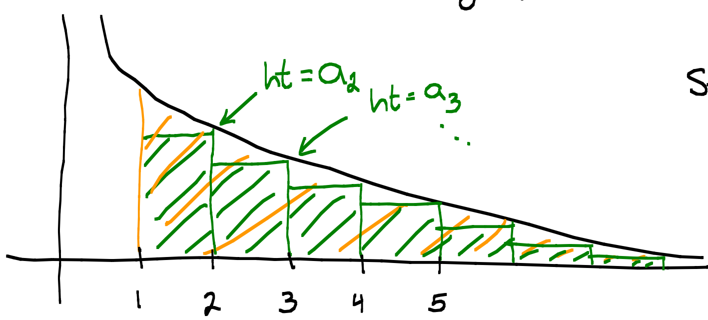
Ex: $\sum_{n=1}^\infty \frac{1}{n^2}$ converges. $f(x) = \frac{1}{x^2}$, and

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(\frac{-1}{t} + \frac{1}{1}\right) = 1 < \infty$$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, so does $\sum \frac{1}{n^2}$.

Also get an estimate on size of $\sum_1^\infty \frac{1}{n^2}$:

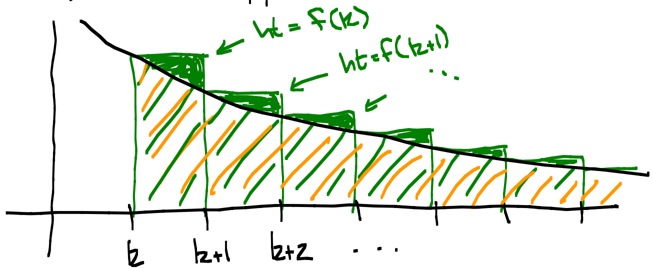
$$\int_1^\infty \frac{1}{x^2}\,dx = \text{area under } y = \frac{1}{x^2} \text{ for } x \geq 1.$$



So $\sum_{n=2}^\infty \frac{1}{n^2} \leq \int_1^\infty \frac{1}{x^2}\,dx = 1$

$\Rightarrow \sum_{n=1}^\infty \frac{1}{n^2} \leq 1 + a_1 = 2.$

For divergence, use upper Riemann sums:



Picture shows that

$$\int_k^\infty f(x)\,dx \leq \sum_{n=k}^\infty f(n)$$

So if $\int_k^\infty f(x)\,dx$ diverges, then it becomes infinite, so $\sum_{n=k}^\infty f(n)$ is infinite.

So we have

$$\sum_{n=k+1}^{\infty} a_n \le \int_k^{\infty} f(x)\,dx \le \sum_{n=k}^{\infty} a_n$$

Ex: $a_n = \dfrac{1}{n\ln(n)}$ $\qquad f(x) = \dfrac{1}{x\ln x}$

Check $\displaystyle\int_2^{\infty} \frac{dx}{x\ln(x)} = \lim_{t\to\infty} \int_2^{t} \frac{dx}{x\ln x}$ $\qquad u = \ln x$, $du = \dfrac{dx}{x}$ $\qquad x=2 \Rightarrow u = \ln 2$, $x = t \Rightarrow u = \ln t$

$$= \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u} = \lim_{t\to\infty} \left(\ln\ln t - \ln\ln 2\right) = \infty \Rightarrow \text{diverges.}$$

Important examples:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\le 1 \end{cases} \longleftrightarrow \int_2^{\infty} \frac{1}{x^p}\,dx \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\le 1 \end{cases}$$

Can use the above inequalities to get estimates of size:

$$\int_k^{\infty} f(x)\,dx \le \sum_{n=k}^{\infty} a_n \le \int_{k-1}^{\infty} f(x)\,dx$$

Ex: $a_n = \dfrac{1}{n^4}$ then

$$\frac{1}{24} = \int_2^{\infty} \frac{1}{x^4}\,dx \le \sum_{n=2}^{\infty} \frac{1}{n^4} \le \int_1^{\infty} \frac{1}{x^4}\,dx = \frac{1}{3}$$

So $1\frac{1}{24} \le \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} \le 1\frac{1}{3}$ $\qquad$ (in fact it equals $\pi^4/90$).

Aside: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann Zeta Function (related to the Riemann Hypothesis)

If s is even, the sum is known (always looks like $\pi^s \cdot$ a fraction)

If s is odd, essentially nothing is known!