Lecture 2: Integration by Parts

Note Title

This is a rule to help us integrate products of functions.

$$\int \underbrace{f(x)}_{u} \cdot \underbrace{g'(x)\,dx}_{dv} = \underbrace{f(x)}_{u} \cdot \underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v} \cdot \underbrace{f'(x)\,dx}_{du}$$

normal names

$$\int u \, dv = u \cdot v - \int v \, du$$

$u = f(x)$
$v = g(x)$

Ex: $\int \underset{\substack{\uparrow \\ u}}{x} \, \underbrace{\sin 2x \, dx}_{dv}$

$u = x \Rightarrow du = dx$

$dv = \sin 2x\,dx \Rightarrow v = -\frac{1}{2}\cos 2x$

$$= x \cdot \left(-\frac{1}{2}\cos 2x\right) - \int \left(-\tfrac{1}{2}\right)\cos 2x \cdot dx$$

$$= \boxed{-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C}$$

Why does this work? Product rule!

$$\frac{d}{dx}\big(f(x)\cdot g(x)\big) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

Integrating both sides & rearranging gives the result.

Ex: $\int \ln x \, dx$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$
$dv = dx \Rightarrow v = x$

$$= (\ln x)\cdot x - \int \cancel{x} \cdot \frac{1}{\cancel{x}}\,dx = \boxed{x \ln x - x + C}$$

How to choose u & dv: LIPET:

hard to $\int$

- **L**ogs
- **I**nverse Trig
- **P**olynomials
- **E**xponentials
- **T**rig

easy to $\int$

Basic Guideline for what to choose for u.

Ex: $\int 2x e^{3x}\,dx$ $\quad u = 2x \Rightarrow du = 2\,dx$, $\quad dv = e^{3x}\,dx \Rightarrow v = \frac{1}{3}e^{3x}$

$$= \frac{2}{3}xe^{3x} - \int \frac{2}{3}e^{3x}\,dx$$

$$= \boxed{\frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + C}$$

Ex: $\int e^x \cos x\,dx$ $\quad u = e^x \Rightarrow du = e^x\,dx$, $\quad dv = \cos x\,dx \Rightarrow v = \sin x$

$$= e^x \sin x - \int e^x \sin x\,dx$$

$u = e^x \Rightarrow du = e^x\,dx$
$dv = \sin x\,dx \Rightarrow v = -\cos x$

$$= e^x \sin x - \left(e^x(-\cos x) - \int e^x(-\cos x)\,dx\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

$$\Rightarrow 2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

$$\Rightarrow \int e^x \cos x\,dx = \boxed{\frac{1}{2}\left(e^x \sin x + e^x \cos x\right)}$$

Integrals of the form $\int e^{ax}\cdot \sin(bx)\,dx$, etc eventually come back to something that has the same form!

Ex: $\int (\ln x)^2\,dx$ $\quad u = (\ln x)^2 \Rightarrow du = \frac{2\ln x}{x}\,dx$, $\quad dv = dx \Rightarrow v = x$

$$= x(\ln x)^2 - \int \cancel{x}\cdot \frac{2\ln x}{\cancel{x}}\,dx$$

$u = \ln x \quad du = \frac{1}{x}\,dx$
$dv = dx \quad v = x$

$$= x(\ln x)^2 - 2\left(x\ln x - \int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx\right)$$

$$= \boxed{x(\ln x)^2 - 2x\ln x + 2x + C}$$

Definite Integration: No change!

$$\int_a^b f(x)\cdot g'(x)\,dx = \left(f(x)\cdot g(x)\right)\Big|_a^b - \int_a^b f'(x)\cdot g(x)\,dx$$

Ex $\displaystyle\int_1^2 x e^x \, dx$

$u = x \Rightarrow du = dx$
$dv = e^x dx \Rightarrow v = e^x$

$$= x e^x \Big|_1^2 - \int_1^2 e^x \, dx$$

$$= (2e^2 - e) - (e^2 - e)$$

$$= e^2$$

Check: $\displaystyle\int x e^x \, dx = xe^x - \int e^x \, dx$

$= xe^x - e^x + C$

So $\displaystyle\int_1^2 x e^x \, dx = (xe^x - e^x)\Big|_1^2 = e^2.$ ✓

Ex: $\displaystyle I = \int_0^{\pi/2} e^{2x} \sin x \, dx$

$u = e^{2x} \quad du = 2e^{2x} dx$
$dv = \sin x \, dx \quad v = -\cos x$

$$= 2e^{2x}(-\cos x)\Big|_0^{\pi/2} - \int_0^{\pi/2} 2e^{2x}(-\cos x)\, dx$$

$u = e^{2x} \quad du = 2e^{2x} dx$
$dv = \cos x \, dx \quad v = \sin x$

$$= \left(2e^{\pi}\cdot 0 - 2(-1)\right) + \left(2e^{2x}\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} 4e^{2x}\sin x \, dx\right)$$

$$= 2 + 2e^{\pi} - 4\int_0^{\pi/2} e^{2x} \sin x \, dx = 2 + 2e^{\pi} - 4I.$$

Solving for I gives: $5 \cdot I = 2 + 2e^{\pi} \Rightarrow I = \boxed{\frac{2}{5} + \frac{2}{5}e^{\pi}}$

Tabular Method: A fast way to integrate $(\text{poly}) \cdot \begin{cases} \text{exp} \\ \text{trig} \end{cases}$

To find $\displaystyle\int p(x) \cdot E(x) \, dx$, write 3 columns:

$p(x)$: polynomial; $E(x)$: trig, exp

diff	integrale	+/−
derivatives of p go here.	Integrals of E go here	−
		+
		−
		+
		⋮

$$\int (2x^2 - 3) e^x \, dx =$$

$$\boxed{(2x^2-3)e^x - (4x)e^x + 4e^x + C}$$

u	dv	+
$2x^2 - 3$	e^x	−
$4x$	e^x	+
4	e^x	−
0	e^x	+

$$\int x^3 \sin\left(x/2\right) dx =$$

$$\boxed{-2x^3 \cos\frac{x}{2} + 12x^2 \sin x/2 + 48x \cos\frac{x}{2} - 96 \sin x/2 + C}$$

u	dv	$+$
x^3	$\sin x/2$	$-$
$3x^2$	$-2\cos x/2$	$+$
$6x$	$-4\sin x/2$	$-$
6	$8\cos x/2$	$+$
0	$16\sin x/2$	$-$