

Lecture 19 - Geometric & Telescoping Series

Note Title

Saw that if

$\sum_{n=0}^{\infty} a_n r^n$ converges, then the limit is $\frac{a_0}{1-r}$
 if $r > 1$, then this diverges.

$$\sum_{n=0}^{\infty} a_n r^n \begin{cases} \text{converges} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

Show this by looking at s_n .

$$S_0 = a_0$$

$$S_1 = a_0 + a_0 r$$

$$S_n = a_0 + a_1 r + \dots + a_n r^n$$

$$r \cdot S_n = \cancel{a_0 r} + \dots + \cancel{a_{n-1} r} + a_n r^{n+1}$$

$$-S_n = \underline{-(a_0 + a_1 r + \dots + a_{n-1} r)}$$

$$\quad\quad\quad -a_0 \qquad\qquad\qquad + a_n r^{n+1}$$

$$(r-1) \cdot s_n = a_0(r^{n+1}-1) \quad \{ \text{if } r \neq 1, \text{ then}$$

$$S_n = \frac{a_0 (1 - r^{n+1})}{1 - r}$$

useful formula for partial sums of a geom series

Series converges if $\lim_{n \rightarrow \infty} s_n$ exists:

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(\frac{a_0}{1-r} - \frac{a_0 r^{n+1}}{1-r} \right) = \frac{a_0}{1-r} - \left(\frac{a_0}{1-r} \lim_{n \rightarrow \infty} r^{n+1} \right)$$

If $|r| < 1$, then $r^{n+1} \rightarrow 0$, so limit exists ; $s_n \rightarrow \frac{a_0}{1-r}$

If $|r| > 1$, then $\lim_{n \rightarrow \infty} r^{n+1}$ does not exist, so $\lim_{n \rightarrow \infty} s_n$ does not exist.

If $r = -1$, then $\lim_{n \rightarrow \infty} r^{n+1} = \cdots = -1$.

$$\text{& if } r=1? \quad S_n = \underbrace{a_0 + a_0r + \dots + a_0r^n}_{(n+1) \text{ terms}} = (n+1) \cdot a_0$$

$\lim_{n \rightarrow \infty} (n+1) a_n$ does not exist, so diverges.

So to determine convergence, we only have to check if $|r| < 1$!

Ex: $\sum_{n=0}^{\infty} 6 \left(\frac{2}{3}\right)^n$ converges. $a_0 = 6$, $r = \frac{2}{3}$

Ex: $\sum_{n=2}^{\infty} \left(\frac{1}{3}\right)^n$ converges. $a_0 = \text{first term} = \frac{1}{9}$
 $r = \text{ratio of } 2^{\text{nd}} \text{ to } 1^{\text{st}} \text{ term}$

$$= \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots = \frac{1}{3}$$

$$= \frac{1/9}{1-1/3} = \boxed{\frac{1}{6}}$$

If our series is geometric (only instances of n are (linear) exponents),
 1^{st} term is always a_0 , $\frac{2^{\text{nd}} \text{ term}}{1^{\text{st}} \text{ term}} = r$.

Ex: $\sum_{n=0}^{\infty} 2^{2n} \cdot 3^{-3n+3} = 27 + 4 + \frac{16}{27} + \dots$
 $a_0 = 27$, $r = \frac{4}{27} < 1$
 $\Rightarrow \text{converges} \quad | \quad \text{sum} = \frac{27}{1 - 4/27} = \frac{729}{23}$

Can also rewrite each term to make it look like $a_0 r^n$:

$$\begin{aligned} 2^{2n} &= (2^2)^n = 4^n \\ 3^{-3n+3} &= 3^3 \cdot (3^{-3})^n = 27 \cdot \left(\frac{1}{27}\right)^n \end{aligned} \quad \left\{ \begin{aligned} 2^{2n} \cdot 3^{-3n+3} &= 27 \cdot \left(\frac{4}{27}\right)^n \\ a_0 &= 27 \\ r &= \frac{4}{27} \end{aligned} \right.$$

Have to remember 3 things:

$$\textcircled{1} \quad a^{b \cdot c} = (a^b)^c$$

$$\textcircled{2} \quad a^b \cdot a^c = a^{b+c}$$

$$\textcircled{3} \quad a^n \cdot b^n = (ab)^n \quad (a \neq b)$$

Ex: $\sum_{n=2}^{\infty} 3^{-2n} 5^{n-2}$: $\left. \begin{aligned} 3^{-2n} &= (3^{-2})^n = \left(\frac{1}{9}\right)^n \\ 5^{n-2} &= 5^{-2} \cdot 5^n = \frac{1}{25} 5^n \end{aligned} \right\} \frac{1}{25} \left(\frac{5}{9}\right)^n$

So first few terms are:

$$\frac{1}{81} + \frac{5}{729} + \dots = \sum_{n=0}^{\infty} \frac{1}{81} \left(\frac{5}{9}\right)^n = \frac{1/81}{1 - 5/9} = \boxed{\frac{1}{36}}$$



One other big convergence example: Telescoping series.

Works if we see terms that will cancel.

Ex: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Partial fractions: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \rightarrow$ let's us identify s_n

$$s_1 = \frac{1}{1} - \frac{1}{2} = 1 - \frac{1}{2}$$

$$s_2 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$s_3 = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

⋮

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

(really: Induction shows $s_n = 1 - \frac{1}{n+1}$: $s_{n-1} = 1 - \frac{1}{n}$, so

$$s_n = s_{n-1} + a_n = \left(1 - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$$\text{So } \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = 1.$$

Ex: $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

Partial fractions: $\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$

$$s_1 = 1 - \frac{1}{3}$$

$$s_2 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) = \left(1 + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{4}\right)$$

$$s_3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) = \left(1 + \frac{1}{2}\right) - \left(\frac{1}{4} + \frac{1}{5}\right)$$

$$s_4 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) = \left(1 + \frac{1}{2}\right) - \left(\frac{1}{5} + \frac{1}{6}\right)$$

$$\vdots \\ s_n = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) = \left(1 + \frac{1}{2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2}\right)$$

$$\text{So } \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{3}{2} - \left(\frac{1}{n+1} + \frac{1}{n+2}\right) = \boxed{\frac{3}{2}} \quad \text{+ series converges}$$

Also works for things like

$$a_n = \sin(n+\pi) - \sin(n), \quad a_n = r^{n+\pi} - r^n, \quad f(n+\pi) - f(n).$$

In these cases, π determines how far we have to go before it telescopes.