Lecture 18 – Sequences & Series

Note Title

Last time: If a_n is bounded and increasing, then a_n converges.

Ex: $\sqrt{2}, \sqrt{2+\sqrt{2}}, \ldots$

a_n defined recursively:

$$a_n = \sqrt{2+a_{n-1}}$$

Then a_n is visibly increasing. Claim: a_n is bounded above by 2:

If $a_{n-1} \le 2$, then $a_n = \sqrt{2+a_{n-1}} \le \sqrt{2+2} = 2.$

$\Rightarrow a_n$ converges & let $L = \lim_{n\to\infty} a_n$. Then taking $\lim_{n\to\infty}$ of $a_n = \sqrt{2+a_{n-1}}$

$$\Rightarrow L = \lim_{n\to\infty} \sqrt{2+a_{n-1}} = \sqrt{2+\lim_{n\to\infty} a_{n-1}} = \sqrt{2+L}$$

$$\Rightarrow L^2 = 2+L \leftrightarrow (L-2)(L+1)=0 \leftrightarrow L=2, \cancel{-1}$$ $a_n \ge 0$

How did we find that this was bounded? Induction

① Show true in the first case $(a_1 = \sqrt{2} < 2)$

② Show how true for $a_n \Rightarrow$ true for a_{n+1} $\left(a_n = \sqrt{2+a_{n-1}} < \sqrt{2+2} = 2\right)$

This is a property of the natural numbers:

every number has a successor $(m \rightsquigarrow m+1 \rightsquigarrow m+2 \rightsquigarrow \ldots)$

So if we know it is true for one point and true for successors, then its true for anything.

Ex: $a_n = 1 + \frac{1}{a_{n-1}}$, $a_1 = 1$. This sequence converges. Let the limit be F:

$$\lim_{n\to\infty} a_n = F = \lim_{n\to\infty}\left(1+\frac{1}{a_{n-1}}\right) = 1 + \frac{1}{\lim_{n\to\infty} a_{n-1}} = 1 + \frac{1}{F}$$

$$\Rightarrow F^2 = F+1 \Rightarrow F^2 - F - 1 = 0$$

$$\Rightarrow F = \frac{1+\sqrt{5}}{2} \text{ or } \cancel{\frac{1-\sqrt{5}}{2}}$$ $a_n \ge 0$ for all n, this is < 0.

Given a sequence a_n, we can form a new sequence:

$$s_n = a_1 + \dots + a_n$$

This is the sequence of partial sums. The limit of this sequence is the infinite series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \dots$

The i in the sum is a "dummy" or "bound" variable. We can use any name: $\sum_{i=1}^{\infty} a_i = \sum_{n=1}^{\infty} a_n = \sum_{*=1}^{\infty} a_*$ etc

In general, it is very hard to find the limit. We can often determine if there is a limit.

Def A series $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} s_n$ exists

If the limit does not exist, the series diverges.

Big Example I: Geometric Series

Let $a_n = a_0 r^n$ $n \geq 0$ & consider

$$\sum_{n=0}^{\infty} a_n = a_0 + a_0 r + a_0 r^2 + \dots$$

Ex: $a_0 = 1$, $r = 1/2$:

$$S_0 = 1$$

$$S_1 = 1 + 1/2 = \frac{3}{2}$$

$$S_2 = 1 + 1/2 + 1/4 = \frac{7}{4}$$

$$S_3 = 1 + 1/2 + 1/4 + 1/8 = \frac{15}{8}$$

$$\vdots$$

$$S_n = S_{n-1} + 1/2^n = \frac{2^{n+1}-1}{2^n} = 2 - 1/2^n$$

So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 - 1/2^n = 2.$ $\Rightarrow$ Series converges &

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2.$$

General case: $a_n = a_0 r^n$

$$S_0 = a_0$$

$$S_1 = a_0 + a_0 r \quad = \quad r \cdot S_0 + a_0$$

$$S_2 = a_0 + \underbrace{a_0 r + a_0 r^2}_{r \cdot S_1} \quad = \quad r \cdot S_1 + a_0$$

$$\vdots$$

$$S_n = a_0 + \underbrace{a_0 r + \dots + a_0 r^n}_{r \cdot S_{n-1}} \quad = \quad r \cdot S_{n-1} + a_0$$

So if this converges, we can find the limit: $\lim_{n \to \infty} S_n = S$

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} (r \cdot S_{n-1} + a_0) = r \cdot \left(\lim_{n \to \infty} S_{n-1} \right) + a_0 \quad = r \cdot S + a_0$$

So $\quad S - rS = a_0 \quad \longleftrightarrow \quad \boxed{S = \frac{a_0}{1-r}}$

See immediately that if $r = 1$, this doesn't work (and $\sum_{n=0}^{\infty} a_0 = a_0 + a_0 + \dots$ diverges)

If $r > 1$, then S and a_0 have opposite signs. The partial sums all have the same sign as $a_0 \Rightarrow$ diverges.