Lecture 18 - Sequences and Series

Last time: If \( a_n \) is bounded and increasing, then \( a_n \) converges.

Ex: \( \sqrt{2}, \sqrt{2+\sqrt{2}}, \ldots \)

\( a_n \) defined recursively:
\[
a_n = \sqrt{2 + a_{n-1}}
\]

Then \( a_n \) is visibly increasing. Claim: \( a_n \) is bounded above by 2:

If \( a_{n-1} \leq 2 \), then \( a_n = \sqrt{2 + a_{n-1}} \leq \sqrt{2 + 2} = 2 \).

\( \Rightarrow \) \( a_n \) converges \( \Rightarrow \) let \( L = \lim_{n \to \infty} a_n \). Then taking \( \lim_{n \to \infty} \) of \( a_n = \sqrt{2 + a_{n-1}} \)

\( \Rightarrow \) \( L = \lim_{n \to \infty} \sqrt{2 + a_{n-1}} = \sqrt{2 + \lim_{n \to \infty} a_{n-1}} = \sqrt{2 + L} \)

\( \Rightarrow \) \( L^2 = 2 + L \leftrightarrow (L-2)(L+1)=0 \leftrightarrow L=2, \not L=-1 \)

How did we find that this was bounded? Induction

1. Show true in the first case \( (a_1 = \sqrt{2} < 2) \)

2. Show how true for \( a_n \) \( \Rightarrow \) true for \( a_{n+1} \) \( (a_n = \sqrt{2 + a_{n-1}} < \sqrt{2 + 2} = 2) \)

This is a property of the natural numbers:

Every number has a successor \( (m \mapsto m+1 \mapsto m+2 \mapsto \ldots) \)

So if we know it is true for one point and true for successors, then it is true for anything.

Ex: \( a_n = 1 + \frac{1}{a_{n-1}}, \ a_1 = 1 \). This sequence converges. Let the limit be \( F \):

\[
\lim_{n \to \infty} a_n = F = \lim_{n \to \infty} \left(1 + \frac{1}{a_{n-1}}\right) = 1 + \frac{1}{\lim_{n \to \infty} a_{n-1}} = 1 + \frac{1}{F}
\]

\( \Rightarrow \) \( F^2 = F + 1 \)

\( \Rightarrow \) \( F^2 - F - 1 = 0 \)

\( \Rightarrow \) \( F = \frac{1 + \sqrt{5}}{2} \) or \( \frac{1 - \sqrt{5}}{2} \)

\( a_n \geq 0 \) for all \( n \), this is \( < 0 \).
Given a sequence \( a_n \), we can form a new sequence:
\[
S_n = a_1 + \ldots + a_n
\]
This is the sequence of partial sums. The limit of this sequence is the **infinite series**
\[
\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \ldots
\]
The \( i \) in the sum is a "dummy" or "bound" variable. We can use any name:
\[
\sum_{i=1}^{n} a_i = \sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} a_k \quad \text{etc}
\]
In general, it is very hard to find the limit. We can often determine if there is a limit.

**Def.** A series \( \sum_{n=1}^{\infty} a_n \) **converges** if \( \lim_{n\to\infty} S_n \) exists.

If the limit does not exist, the series **diverges**.

**Big Example I: Geometric Series**

Let \( a_n = a_0 r^n \quad n \geq 0 \) consider:
\[
\sum_{n=0}^{\infty} a_n = a_0 + a_0 r + a_0 r^2 + \ldots
\]

**Ex:** \( a_0 = 1 \), \( r = \frac{1}{2} \):

\[
S_0 = 1
\]
\[
S_1 = 1 + \frac{1}{2} = \frac{3}{2}
\]
\[
S_2 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}
\]
\[
S_3 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}
\]
\[
\vdots
\]
\[
S_n = S_{n-1} + \frac{1}{2^n} = \frac{2^{n+1} - 1}{2^n} = 2 - \frac{1}{2^n}
\]

So \( \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 - \frac{1}{2^n} = 2. \Rightarrow \text{ series converges } \)

\[
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2.
\]
General case: \( a_n = a_0 r^n \)

\[
S_0 = a_0 \\
S_1 = a_0 + a_0 r = r \cdot S_0 + a_0 \\
S_2 = a_0 + a_0 r + a_0 r^2 = r \cdot S_1 + a_0 \\
\vdots \\
S_n = a_0 + a_0 r + \cdots + a_0 r^n = r \cdot S_{n-1} + a_0 \\
\frac{r \cdot S_{n-1}}{r \cdot S_{n-1}}
\]

So if this converges, we can find the limit: \( \lim_{n \to \infty} S_n = S \)

\[
S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} (r \cdot S_{n-1} + a_0) = r \left( \lim_{n \to \infty} S_{n-1} \right) + a_0 = r \cdot S + a_0
\]

So \( S - rS = a_0 \) \( \iff \) \[
S = \frac{a_0}{1-r}
\]

See immediately that if \( r = 1 \), this doesn't work (and \( \sum_{n=0}^{\infty} a_n = a_0 + a_0 + \cdots \) diverges)

If \( r > 1 \), then \( S \) and \( a_0 \) have opposite signs. The partial sums all have the same sign as \( a_0 \) \( \Rightarrow \) diverges.