Lecture 17 - Sequences

Def: A **sequence** is a collection of real numbers, each following the next.

Ex: 0, 1, 2, 3, 4, ...

- 0, 2, 4, 6, 8, ...
- 3, 3.1, 3.14, 3.141, ...

Def: The name of a particular element is its **index**.

We can start our index at any integer. This determines the form of the general element.

Ex:

\[ a_n: 1, 2, 3, \ldots \]

1. Start \( n \): 0, 1, 2, \( \Rightarrow a_n = n + 1 \)
2. Start \( n \): 1, 2, 3, \( \Rightarrow a_n = n \)
3. Start \( n \): 2, 3, 4, \( \Rightarrow a_n = n - 1 \)

Ex:

\[ a_n: 0, 2, 4, 6, \ldots \]

- 0: 0, 1, 2, 3, \( \Rightarrow a_n = 2n \)
- 1: 1, 2, 3, 4, \( \Rightarrow a_n = 2(n-1) \)
- 5: 5, 6, 7, 8, \( \Rightarrow a_n = 2(n-5) \)

Ex:

\[ a_n: 1, -3, 5, -7, \ldots \]

- 0: 0, 1, 2, 3, \( a_n = (-1)^n (2n+1) \)
- 1: 1, 2, 3, 4, \( a_n = (-1)^{n-1} (2(n-1)+1) = (-1)^n (2n-1) \)

Most important concept is convergence.

Def: A sequence \( a_n \) **converges** to \( L \) if for every \( \epsilon > 0 \), there is a big \( N \) s.t. \( |a_n - L| < \epsilon \) for all \( n > N \).

What does this mean? \( |a_n - L| < \epsilon \leftrightarrow a_n \) in the interval \((L - \epsilon, L + \epsilon)\)
If \( a_n \) converges to \( L \), we write \( \lim_{n \to \infty} a_n = L \) or \( a_n \to L \).

So the definition means that given any small interval about \( L \), the terms of the sequence eventually all land in the interval.

**Example:** \( a_n = 1 - \frac{1}{n} \) this converges to 1.

So given \( \varepsilon > 0 \), we can find an \( N \) s.t. \( \frac{1}{N} < \varepsilon \) for all \( n > N \) (just take \( N > \frac{1}{\varepsilon} \)).

Another way to find this: let \( f(x) = 1 - \frac{1}{x} \). Then \( a_n = f(n) \).

Then \( \lim_{n \to \infty} a_n = \lim_{x \to \infty} f(x) = 1 \).

This is a general fact:

**If** \( f(n) = a_n \) and \( \lim_{x \to \infty} f(x) = L \), then \( \lim_{n \to \infty} a_n = L \).

We normally use this to find convergence of a sequence.

**Example:** \( a_n = \frac{n + 3n}{n^2 + 1} = f(n) \) where \( f(x) = \frac{\ln(x)}{x} \).

Then \( \lim_{n \to \infty} a_n = \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\ln(x)}{x} = 0 \).

L'Hôpital's rule.

\[ b_n = \frac{n^3 + 3n}{2n^3 + 1} = f(n) \] where \( f(x) = \frac{x^3 + 3x}{2x^3 + 1} \).

Then \( \lim_{n \to \infty} b_n = \lim_{x \to \infty} \frac{x^3 + 3x}{2x^3 + 1} = \lim_{x \to \infty} \frac{1 + \frac{3}{x^2}}{2 + \frac{1}{x^3}} = \frac{1}{2} \).

2. Other approaches:

   I. Squeeze Theorem
If \( a_n \leq b_n \leq c_n \) for all \( n \), then \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c = L \), then \( \lim_{n \to \infty} b_n = L \).

\[
\begin{align*}
\text{Ex} & \quad b_n = \frac{1}{n} \sin(n) \quad \text{Then} \quad a_n = \frac{-1}{n}, \quad c_n = \frac{1}{n} \quad \text{has} \\
& \quad a_n \leq b_n \leq c_n \quad \text{for all} \quad n. \\
& \quad \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = 0 \quad \Rightarrow \quad \lim_{n \to \infty} b_n = 0.
\end{align*}
\]

II. Need some def's:

**Def:** A sequence is **bounded above** if there is an \( M \) s.t. \( a_n \leq M \) for all \( n \).

**Ex:** \( a_n = 1 + \frac{1}{n} \) then \( a_n \leq 2 \) for all \( n \).

**Def:** A sequence is **bounded below** if there is an \( m \) s.t. \( a_n \geq m \) for all \( n \).

**Ex:** \( a_n = 1 + \frac{1}{n} \) then \( a_n \geq 0 \) for all \( n \).

(A sequence is **bounded** if it is bounded above \& below)

**Def** A sequence is **increasing** if \( a_n \leq a_{n+1} \) for all \( n \).

A sequence is **decreasing** if \( a_n \geq a_{n+1} \) for all \( n \).

**Ex:** \( a_n = 1 + \frac{1}{n} \) is decreasing.

\( b_n = 1 - \frac{1}{n} \) is increasing.

**Thm** If \( a_n \) is increasing and bounded above, then \( a_n \) converges.

If \( b_n \) is decreasing and bounded below, then \( b_n \) converges.

This uses a key property of the real numbers: Least Upper Bounds.

Every bounded set has a least upper bound.