Lecture 17 – Sequences

Note Title

Def A sequence is a collection of real numbers, each following the next.

Ex: ① 1, 2, 3, 4, ...

② 0, 2, 4, 6, 8, ...

③ 3, 3.1, 3.14, 3.141, ...

④ $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$

⑤ $1+\;\; 1+\frac{1}{1+1},\;\; 1+\frac{1}{1+\frac{1}{1+1}}, \ldots$

Def: The name of a particular element is its index.

We can start our index at any integer. This determines the form of the general element.

Ex a_n: 1, 2, 3, ...

start @ 0: 0 1 2 $\Rightarrow$ $a_n = n+1$

start @ 1: 1 2 3 $\Rightarrow$ $a_n = n$

start @ 2: 2 3 4 $\Rightarrow$ $a_n = n-1$

Ex: a_n: 0, 2, 4, 6, ...

@ 0: 0 1 2 3 $\Rightarrow$ $a_n = 2n$

@ 1: 1 2 3 4 $\Rightarrow$ $a_n = 2(n-1)$

@ 5: 5 6 7 8 $\Rightarrow$ $a_n = 2(n-5)$

doesn't matter where you start your index

Ex: $a_n = 1, -3, 5, -7, \ldots$

@ 0: 0 1 2 3 $\quad a_n = (-1)^n(2n+1)$

@ 1: 1 2 3 4 $\quad a_n = (-1)^{n-1}(2(n-1)+1) = (-1)^{n-1}(2n-1)$

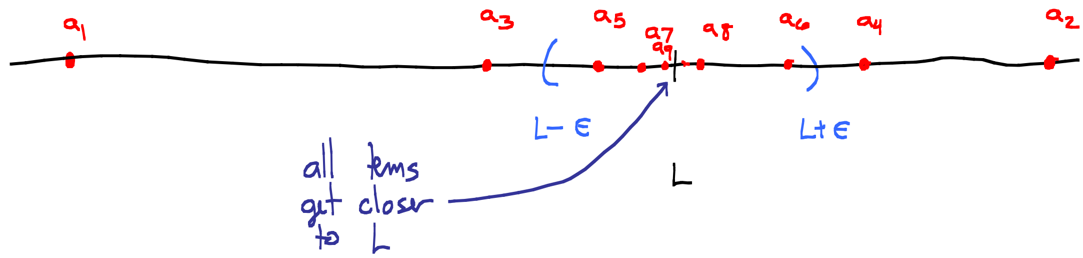
Most important concept is convergence.

Def A sequence a_n converges to L if for every $\epsilon>0$ ("error"), there is an N (big #) s.t. $|a_n - L| < \epsilon$ for all $n > N$.

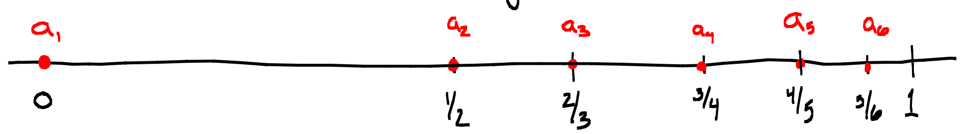
What does this mean? $|a_n - L| < \epsilon \longleftrightarrow a_n$ in the interval $(L-\epsilon, L+\epsilon)$

If a_n converges to L, we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$

So the definition means that given any small interval about L, the terms of the sequence eventually all land in the interval.



Ex $\quad a_n = 1 - \frac{1}{n}$ $\quad$ this converges to 1.



So given $\epsilon > 0$, we can find an N s.t. $\frac{1}{n} < \epsilon$ for all $n > N$ (just take $N > \frac{1}{\epsilon}$).

Another way to find this: let $f(x) = 1 - \frac{1}{x}$. Then $a_n = f(n)$.

Then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = 1.$

This is a general fact:

If $f(n) = a_n$ and $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

We normally use this to find convergence of a sequence.

Ex $\quad a_n = \dfrac{\ln(n)}{n} = f(n)$ where $f(x) = \dfrac{\ln(x)}{x}$

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{\ln(x)}{x} \underset{\uparrow}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

L'Hôpital's rule

$$b_n = \frac{n^3+3n}{2n^3+1} = f(n) \quad \text{where} \quad f(x) = \frac{x^3+3x}{2x^3+1}$$

$$\lim_{n\to\infty} b_n = \lim_{x\to\infty} \frac{x^3+3x}{2x^3+1} = \lim_{x\to\infty} \frac{1+3/x^2}{2+1/x^3} = \frac{1}{2}.$$

2 Other approaches:

I. Squeeze Theorem

If $a_n \le b_n \le c_n$ for all n & $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c = L$, then

$$\lim_{n\to\infty} b_n = L.$$

Ex $b_n = \frac{1}{n}\sin(n)$ Then $a_n = -1/n$, $c_n = 1/n$ has

$$a_n \le b_n \le c_n \quad \text{for all } n.$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0 \Rightarrow \lim_{n\to\infty} b_n = 0.$$

II. Need some defs:

Def: A sequence is bounded above if there is an M ("Max") s.t.

$$a_n \le M \quad \text{for all } n.$$

Ex: $a_n = 1 + \frac{1}{n}$ then $a_n \le 2$ for all n

Def: A sequence is bounded below if there is an m ("min") s.t.

$$a_n \ge m \quad \text{for all } n.$$

Ex: $a_n = 1 + \frac{1}{n}$ then $a_n \ge 0$ for all n.

(A sequence is bounded if it is bounded above & below)

Def A sequence is increasing if $a_n \le a_{n+1}$ for all n.

A sequence is decreasing if $a_n \ge a_{n+1}$ for all n.

Ex: $a_n = 1 + \frac{1}{n}$ is decreasing.

$b_n = 1 - \frac{1}{n}$ is increasing.

Thm If a_n is increasing and bounded above, then a_n converges

If b_n is decreasing and bounded below, then b_n converges.

This uses a key property of the real numbers: Least Upper Bounds

Every bounded set has a least upper bound.