Lecture 16 - General Regions | Polar Area

Recall: Type I: \( a \leq x \leq b, c(x) \leq y \leq d(x) \)

Type II: \( c \leq y \leq d, a(y) \leq x \leq b(y) \)

If \( R \) is both type I and type II, we get to choose which one to use.

If a region is neither type I nor type II, have to split things up.

Picture:

\[ \iint_R f(x,y) \, dA = \iint_{R_1} f(x,y) \, dA + \iint_{R_2} f(x,y) \, dA \]

Sometimes called "subadditivity". This means we can always break up a region into ones of Type I or II.

(Caveat: \( R_1 \) and \( R_2 \) must hit nicely ie only along boundary. This ensures that there is no volume from their intersection)

Ex: \[
\begin{array}{c}
R_1 \\
R_2
\end{array}
\]

Neither type I nor type II. But!

\( R_1 \) is type I. \( R_2 \) is type II.

Can use our previous discussion to find these

Ex: \[
\begin{array}{c}
R \quad (x+1+y) \leq 1
\end{array}
\]

Set-up \( \iint_R 2 \, dA \):

\[
R_1: \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1-x \\
R_2: \quad 0 \leq x \leq 1, \quad x-1 \leq y \leq 0 \\
R_3: \quad -1 \leq y \leq 0, \quad -1-y \leq x \leq 0 \\
R_4: \quad 0 \leq y \leq 1, \quad y-1 \leq x \leq 0
\]

All are Type I II so any works
\[ = \int_0^1 \int_{-x}^x 2 \, dy \, dx + \int_0^1 \int_{-y}^0 2 \, dy \, dx + \int_{-1}^0 \int_{-1-y}^0 2 \, dx \, dy + \int_0^1 \int_{y-1}^0 2 \, dx \, dy \]

Fubini's Theorem sometimes lets us solve things we couldn't otherwise:

1. Write iterated integral as \( \iint dA \)
2. Write \( \iint dA \) as an iterated integral in the other way.

Ex: Look at \( \int \int \ e^{x^3} \, dA \) when \( R = \begin{array}{c} \frac{y^2}{2} \\ 0 \end{array} \). Have 2 set-ups:

**R Type I:**
\[ = \int_0^1 \int_0^{x^2} e^{x^3} \, dy \, dx = \int_0^1 ye^x \Big|_{y=0}^{y=x^2} \, dx = \int_0^1 3x^2 e^x \, dx = \int_0^1 e^u \, du = e - 1 \]

**R Type II:**
\[ = \int_0^1 \int_0^{\sqrt[3]{3}} e^{x^3} \, dx \, dy \] (Can't integrate \( e^{x^3} \))

Ex:
\[ \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \sin(x^2) \, dx \, dy \]

\[ = \int_0^{\frac{\pi}{4}} \int_0^{\sin(x^2)} y \sin(x^2) \, dy \, dx = \int_0^{\frac{\pi}{4}} x \sin(x^2) \, dx \quad u = x^2 \quad du = 2x \, dx \]

\[ = \frac{1}{2} \int_0^{\frac{\pi}{4}} \sin u \, du = -\frac{1}{2} \cos u \bigg|_{u=0}^{u=\frac{\pi}{4}} = \frac{1}{2} \]

Polar Area:

Can do with double integrals. Need a formula (derived below):

\[ dA = r \, dr \, d\theta \]

Can remember with units:
- \( dA \leftrightarrow \text{area} \leftrightarrow \text{length}^2 \)
- \( dr \leftrightarrow \text{length} \leftrightarrow \text{length} \)
- \( d\theta \leftrightarrow \text{angle} \leftrightarrow \text{no units} \)

So need an additional length.
Only use one example: polar type I: \( \alpha \leq \theta \leq \beta \quad r(g(\theta)) = g(\theta) \)

\[
\text{Area: } \int_\alpha^\beta \int_r^g(\theta) r \, dr \, d\theta = \int_\alpha^\beta \frac{1}{2} r^2 \bigg| _r = f(\theta) d\theta = \\
\int_\alpha^\beta \frac{1}{a^2} \left( g(\theta) \right)^2 - \frac{1}{a^2} \left( f(\theta) \right)^2 \, d\theta
\]

Ex: \( r(\theta) = 1 + \cos \theta \quad 0 \leq \theta \leq \pi \)

Outer radius: \( 1 + \cos \theta \)

Inner radius: \( 0 \)

\[\Rightarrow \text{area} = \int_0^\pi \frac{1}{a^2} \left( 1 + \cos \theta \right)^2 - \frac{1}{a^2} (0)^2 \, d\theta = \int_0^\pi \frac{1}{a^2} (1 + 2 \cos \theta + \cos^2 \theta) \, d\theta = \int_0^\pi \frac{1}{2} \left( \frac{3}{4} + \frac{1}{2} \cos \theta \right) \, d\theta = \frac{3}{4} \theta + \frac{1}{8} \sin 2\theta \bigg|_0^\pi = \frac{3\pi}{4} = \frac{3\pi}{4} \]

Ex: Area inside \( r = 2 \sin \theta \) 
Outside \( r = 1 \) 

These hit at \( \theta = \frac{\pi}{6}, \, 5\pi/6 \)

\[\Rightarrow \text{Area} = \int_{\pi/6}^{5\pi/6} \frac{1}{2} \left( 2 \sin \theta \right)^2 - \frac{1}{2} (1)^2 \, d\theta = \left( \frac{1}{2} \theta - \frac{1}{2} \sin 2\theta \right) \bigg|_{\pi/6}^{5\pi/6} = \frac{\pi}{3} - \frac{1}{2} \sin \left( \frac{5\pi}{3} \right) + \frac{1}{2} \sin \left( \frac{\pi}{3} \right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \]
Double Integrals in Polar

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

Find an expression for \( dA \).

**Rectangular:**

\[ dA = dx \, dy \]

\[ dA = dxdy \]

**Polar:**

\[ r = \alpha \text{ is a circle} \]
\[ \theta = \alpha \text{ is a ray} \]

\[ dA = \text{area of red piece}. \]

Need: The area of a sector with angle \( \theta \) and radius \( R \) is

\[ \frac{1}{2} R^2 \theta \]

So red area:

\[ \left( \frac{1}{2} R^2 \theta \right) - \left( \frac{1}{2} r^2 \theta \right) = \frac{1}{2} (R^2 - r^2) \, d\theta \]

outside area inside area

\[ = \frac{1}{2} (R+r)(R-r) \, d\theta \]

\[ R-r = \Delta r \approx dr, \text{ so if this is small, } R+r = (r+\Delta r) + r = 2r + \Delta r \approx 2r \]

\[ \Rightarrow \Delta A = \frac{1}{2} (\Delta r) \, dr \, d\theta \Rightarrow \]

\[ dA = r \, dr \, d\theta \]