Lecture 16 - General Regions & Polar area

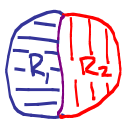
Note Title

Recall: Type I: $a \le x \le b \quad c(x) \le y \le d(x)$

Type II: $c \le y \le d \quad a(y) \le x \le b(y)$

If R is both type I & type II, we get to choose which one to use.

If a region is neither type I nor type II, have to split things up.

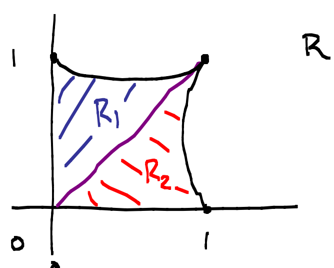
Picture: 

R_1 and R_2 intersect only along their boundary, and let R be all of the points in either R_1 or R_2.

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

Sometimes called "subadditivity". This means we can always break up a region into ones of Type I or II.

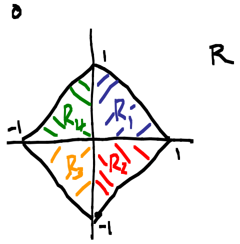
(Caveat: R_1 and R_2 must hit nicely ie only along boundary. This ensures that there is no volume from their intersection)

Ex: 

Neither type I nor type II. But!

R_1 is type I.
R_2 is type II.

} Can use our previous discussion to find these

Ex: 

($\Leftrightarrow \quad |x|+|y| \le 1$)

Set-up $\iint_R 2\,dA$:

R_1: $0 \le x \le 1, \quad 0 \le y \le 1-x$

R_2: $0 \le x \le 1, \quad x-1 \le y \le 0$

R_3: $-1 \le y \le 0, \quad -1-y \le x \le 0$

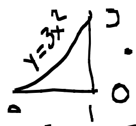
R_4: $0 \le y \le 1, \quad y-1 \le x \le 0$

} all are Type I & II so any works

$$= \int_0^1\int_0^{1-x} 2\,dy\,dx + \int_0^1\int_{x-1}^{0} 2\,dy\,dx + \int_{-1}^0\int_{-1-y}^{0} 2\,dx\,dy + \int_0^1\int_{y-1}^{0} 2\,dx\,dy$$

Fubini's Theorem sometimes lets us solve things we couldn't otherwise:

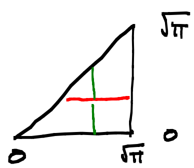
1. Write iterated integral as $\iint_R dA$
2. Write $\iint_R dA$ as an iterated integral in the other way.

Ex Look at $\iint_R e^{x^3}\,dA$ when $R =$ . Have 2 set-ups:

R Type I: $$= \int_0^1\int_0^{3x^2} e^{x^3}\,dy\,dx = \int_0^1 y e^{x^3}\Big|_0^{3x^2} dx = \int_0^1 3x^2 e^{x^3}\,dx = \int_0^1 e^u\,du = e-1$$

R Type II: $$= \int_0^3\int_{\sqrt{y/3}}^{1} e^{x^3}\,dx\,dy = \;?$$ (Can't integrate e^{x^3})

Ex $$\int_0^{\sqrt{\pi}}\int_y^{\sqrt{\pi}} \sin(x^2)\,dx\,dy$$
can't do



$$= \int_0^{\sqrt{\pi}}\int_0^{x} \sin(x^2)\,dy\,dx = \int_0^{\sqrt{\pi}} y\sin(x^2)\Big|_{y=0}^{y=x} dx = \int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$$

$u = x^2$
$du = 2x\,dx$

$$= \frac{1}{2}\int_0^{\pi} \sin u\,du = -\frac{1}{2}\cos u\Big|_{u=0}^{u=\pi} = \boxed{1}$$

Polar Area:

Can do with double integrals. Need a formula (derived below):

$$dA = r\,dr\,d\theta$$

Can remember with units:

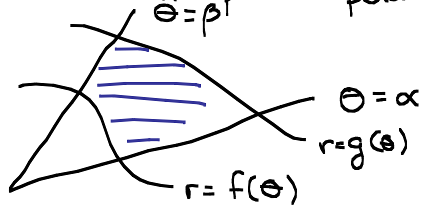
$dA \longleftrightarrow$ area $\longleftrightarrow$ length^2

$dr \longleftrightarrow$ length $\longleftrightarrow$ length

$d\theta \longleftrightarrow$ angle $\longleftrightarrow$ (no units)

So need an additional length.

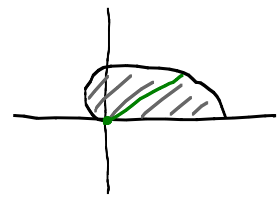
Only use one example: polar type I: $\alpha \le \theta \le \beta$ $\quad f(\theta) \le r \le g(\theta)$



<u>Area</u>: $\displaystyle\int_\alpha^\beta \int_{f(\theta)}^{g(\theta)} r \, dr \, d\theta = \int_\alpha^\beta \frac{1}{2} r^2 \Big|_{r=f(\theta)}^{r=g(\theta)} d\theta =$

$$\int_\alpha^\beta \frac{1}{2}\underbrace{(g(\theta))^2}_{\text{outer radius}} - \frac{1}{2}\underbrace{(f(\theta))^2}_{\text{inner radius}} \, d\theta$$

<u>Ex</u>: $r(\theta) = 1 + \cos\theta \qquad 0 \le \theta \le \pi$

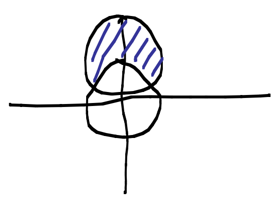


outer radius: $1 + \cos\theta$
inner radius: 0

$$\Rightarrow \quad \text{area} = \int_0^\pi \frac{1}{2}(1+\cos\theta)^2 - \frac{1}{2}(0)^2 \, d\theta = \int_0^\pi \frac{1}{2}(1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^\pi \frac{1}{2}\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta = \frac{3}{4}\theta + \sin\theta + \frac{1}{8}\sin 2\theta \Big|_0^\pi = \boxed{\frac{3\pi}{4}}$$

<u>Ex</u>: Area inside $r = 2\sin\theta \longleftrightarrow x^2 + (y-1)^2 = 1$ (why?)
& outside $r = 1 \longleftrightarrow x^2 + y^2 = 1$



These hit at $\theta = \pi/6,\ 5\pi/6$

$$\Rightarrow \text{Area} = \int_{\pi/6}^{5\pi/6} \frac{1}{2}(2\sin\theta)^2 - \frac{1}{2}(1)^2 \, d\theta$$

$$= \int_{\pi/6}^{5\pi/6} (1 - \cos 2\theta) - \frac{1}{2} \, d\theta = \left(\frac{1}{2}\theta - \frac{1}{2}\sin 2\theta\right)\Big|_{\pi/6}^{5\pi/6} = \frac{\pi}{3} - \frac{1}{2}\underset{\shortparallel}{\sin(5\pi/3)} + \frac{1}{2}\sin(\pi/3)$$

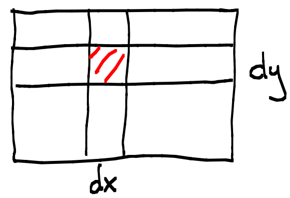
$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

Double Integrals in Polar

$x = r\cos\theta$
$y = r\sin\theta$

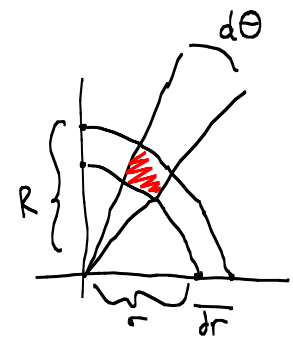
Find an expression for dA.

Rectangular:



$dA = dx\,dy$

Polar: $r = a$ is a circle $\quad\theta = \alpha$ is a ray $\Big\} \Rightarrow$



dA = area of red piece.

Need: The area of a sector w/ angle θ and radius R is $\frac{1}{2}R^2\theta$

So red area: $\underbrace{\left(\tfrac{1}{2}R^2 d\theta\right)}_{\text{outer area}} - \underbrace{\left(\tfrac{1}{2}r^2 d\theta\right)}_{\text{inner area}} = \frac{1}{2}(R^2 - r^2)\,d\theta$

$= \frac{1}{2}(R+r)(R-r)\,d\theta$

$R - r = \Delta r \approx dr$, so if this is small, $R + r = (r + \Delta r) + r = 2r + \Delta r \approx 2r$

$\Rightarrow \Delta A \quad \frac{1}{2}(2r)\,dr\,d\theta \Rightarrow$

$dA = r\,dr\,d\theta$