Lecture 15 - Iterated Integrals

Note Title

very hard to compute

<u>Ex</u> $f(x,y) = 1$, $R = [0,1]\times[0,1]$ Then $\sum f(x_i^*, y_j^*)\,\Delta A = \sum \Delta A$

Now $\sum \Delta A$ = sum of areas of rectangles that cover $[0,1]\times[0,1]$

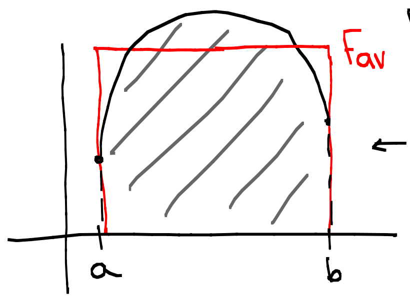
= area of $[0,1]\times[0,1] = 1$.

In fact, for any R,

$$\iint_R dA = \text{area}(R)$$

Can also look at average values:

f_{av} = value so that if we have one box with this height and base R, we get the right volume.



$\leftarrow$ 1 var case.

So $f_{av}\cdot \text{area}(R) = \iint_R f(x,y)\,dA$.

We compute these with a "fundamental theorem".

<u>Thm</u> If f is continuous, then

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy\right)dx$$

hold x constant

hold y constant

$$= \int_c^d \left(\int_a^b f(x,y)\,dx\right)dy$$

<u>Ex</u>: $R = [0,2]\times[0,2]$ $f(x,y) = xy$

$$\iint_R xy\,dA = \int_0^2 \underbrace{\int_0^2 xy\,dy}\,dx$$

$$\int_0^2 \left(\frac{xy^2}{2}\right)\Big|_{y=0}^{y=2} dx = \int_0^2 2x\,dx = \boxed{4}$$

Ex: $R = [0, \pi/2] \times [0, \pi/4]$ $\quad f(x,y) = \sin x + \cos y$

$$\iint_R f(x,y)\, dA = \int_0^{\pi/2} \underbrace{\int_0^{\pi/4} \sin x + \cos y \; dy}\, dx$$

$$\int_0^{\pi/2} \left(y \sin x + \sin y \right) \Big]_{y=0}^{y=\pi/4} dx = \int_0^{\pi/2} \frac{\pi}{4} \sin x + \frac{\sqrt{2}}{2} \; dx$$

$$= -\frac{\pi}{4} \cos x + x\frac{\sqrt{2}}{2} \Big]_{x=0}^{\pi/2} = \boxed{\frac{\pi}{4} + \frac{\pi\sqrt{2}}{4}}$$

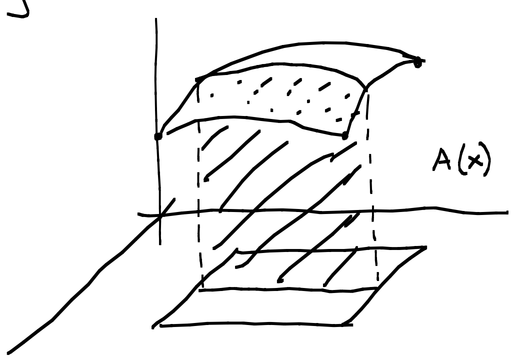
Fubini's Theorem If f is continuous, then

$$\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

In other words, the order of integration doesn't matter.

Why do we have such a formula? Compare with cross-sections.

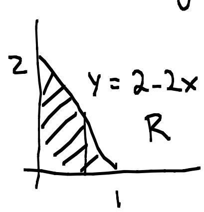
$A(x) = \int_c^d f(x,y)\, dy$ = area of cross-section $\parallel$ to (y,z)-plane @ x:



So formula for volume by cross-sectional area is

$$\iint_R f(x,y)\, dA = V = \int_a^b A(x)\, dx = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

What about non-rectangular R? Express one variable's bounds in terms of the other:



; $f(x,y) = 3x$

So $0 \leq x \leq 1$ and for fixed x,

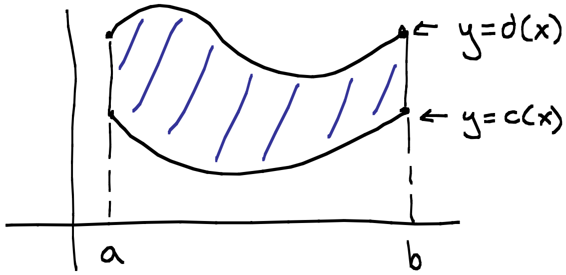
$0 \leq y \leq 2-2x$.

So $$\iint_R f(x,y)\, dA = \int_0^1 \int_0^{2-2x} 3x\, dy\, dx = \int_0^1 (3xy)\Big]_0^{2-2x} dx = \int_0^1 6x - 6x^2\, dx$$

$$= 3x^2 - 2x^3 \Big]_0^1 = \boxed{1}$$

2 Big Cases:

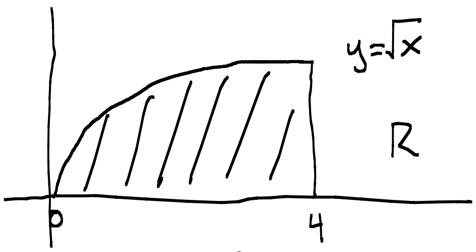
I. Bound by 2 functions of x : $a \le x \le b$, $c(x) \le y \le d(x)$



y a function of x $\Rightarrow$ integrate w.r.t. y first!

$$\iint_R F(x,y)\, dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dy\ dx$$

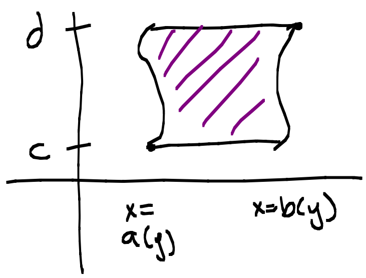
Ex:



$$\iint_R yx\, dA = \int_0^4 \int_0^{\sqrt{x}} yx\ dy\ dx$$

$$= \int_0^4 \frac{xy^2}{2}\Big|_{y=0}^{y=\sqrt{x}} dx = \int_0^4 \frac{x^2}{2}\, dx = \frac{x^3}{6}\Big|_0^4 = \boxed{\frac{32}{3}}$$

II. Bound by 2 functions of y :

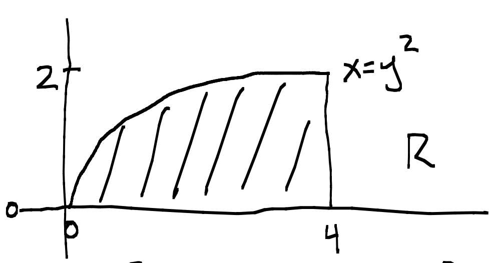


$c \le y \le d$, $a(y) \le x \le b(y)$

x a function of y $\Rightarrow$ integrate w.r.t. x first!

$$\iint_R F(x,y)\, dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\, dx\, dy$$

Ex:



$$\iint_R yx\, dA = \int_0^2 \int_{y^2}^4 yx\ dy\ dx$$

$$= \int_0^2 \frac{yx^2}{2}\Big|_{y^2}^4 dy = \int_0^2 8y - \frac{y^5}{2}\, dy = \left(4y^2 - \frac{y^6}{12}\right)\Big|_0^2 = \boxed{32/3}$$

must be other answer because R and f are unchanged!