very hard to compute

Ex: \( f(x,y) = 1 \), \( R = [0,1] \times [0,1] \) Then \( \sum f(x^*, y^*) \Delta A = \sum \Delta A \)

Now \( \sum \Delta A = \) sum of areas of rectangles that cover \( [0,1] \times [0,1] \)

\( = \) area of \( [0,1] \times [0,1] = 1. \)

In fact, for any \( R \),

\[ \iint_R dA = \text{area}(R) \]

Can also look at average values:

\( f_{av} = \) value so that if we have one box with this height and base \( R \), we get the right volume.

\[ f_{av} \cdot \text{area}(R) = \iint_R f(x,y) \, dA. \]

We compute these with a "fundamental theorem".

**Theorem:** If \( f \) is continuous, then

\[ \iint_R F(x,y) \, dA = \int_a^b \left( \int_c^d f(x,y) \, dy \right) \, dx \]

\[ \text{hold } x \text{ constant} \]

\[ \text{hold } y \text{ constant} \]

\[ = \int_c^d \left( \int_a^b f(x,y) \, dx \right) \, dy \]

**Ex:** \( R = [0,2] \times [0,2] \) \( f(x,y) = xy \)

\[ \iint_R xy \, dA = \int_0^2 \int_0^2 xy \, dy \, dx \]

\[ \int_0^2 \left( \frac{x^2 y^2}{2} \right) \bigg|_{y=0}^{y=2} \, dx = \int_0^2 2x \, dx = 4 \]
Ex: \( R = [0, \pi/2] \times [0, \pi/4] \)

\[ f(x, y) = \sin x + \cos y \]

\[
\iint_{R} f(x, y) \, dA = \int_{0}^{\pi/2} \int_{0}^{\pi/4} \sin x + \cos y \, dy \, dx
\]

\[
\int_{0}^{\pi/2} \left( \int_{0}^{\pi/4} (\sin x + \cos y) \, dy \right) \, dx = \int_{0}^{\pi/2} \left( \int_{0}^{\pi/4} \sin x + \frac{\sqrt{2}}{2} \, dy \right) \, dx
\]

\[
= -\frac{\pi}{4} \cos x + \pi \int_{0}^{\pi/2} \frac{\sqrt{2}}{2} \, dx = \frac{\pi}{4} + \frac{\pi \sqrt{2}}{4}
\]

**Fubini’s Theorem.** If \( f \) is continuous, then

\[
\int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy.
\]

In other words, the order of integration doesn’t matter.

Why do we have such a formula? Compare with cross-sections.

\[ A(x) = \int_{c}^{d} f(x, y) \, dy \quad \text{area of cross-section \( \parallel \) to \( (y, z) \)-plane \( \sigma \) \( x \):} \]

\[
\Rightarrow \int_{1}^{6} \int_{0}^{1} f(x, y) \, dA = \int_{0}^{1} A(x) \, dx = \int_{0}^{1} \left( \int_{0}^{1} f(x, y) \, dy \right) \, dx
\]

What about non-rectangular \( R \)? Express one variable’s bounds in terms of the other:

\[
\begin{align*}
R & : \begin{cases} y = 2 - 2x \\ y = 0 \end{cases} \\
& : 0 \leq x \leq 1
\end{align*}
\]

\[ f(x, y) = 3x \]

So \( 0 \leq x \leq 1 \) and for fixed \( x \), \( 0 \leq y \leq 2 - 2x \).

\[
\int_{0}^{1} \int_{0}^{2-2x} 3x \, dy \, dx = \int_{0}^{1} (3xy) \bigg|_{0}^{2-2x} \, dx = \int_{0}^{1} (6x - 6x^2) \, dx
\]

\[
= 3x^2 - 2x^3 \bigg|_{0}^{1} = 1
\]
2. Big Cases:

I. Bound by 2 functions of \( x \): \( a \leq x \leq b \), \( c(x) \leq y \leq d(x) \)

\[
\iint_{R} F(x, y) \, dA = \int_{a}^{b} \int_{c(x)}^{d(x)} f(x, y) \, dy \, dx
\]

Ex:

\[
\iint_{R} xy \, dA = \int_{0}^{4} \int_{0}^{\sqrt{x}} xy \, dy \, dx
\]

\[
= \int_{0}^{4} \frac{x^{2}}{2} \, dx = \int_{0}^{4} \frac{x^{2}}{2} \, dx = \frac{x^{3}}{6} \bigg|_{0}^{4} = \frac{32}{3}
\]

II. Bound by 2 functions of \( y \):

\[
\iint_{R} F(x, y) \, dA = \int_{c}^{d} \int_{a(y)}^{b(y)} f(x, y) \, dx \, dy
\]

Ex:

\[
\iint_{R} xy \, dA = \int_{0}^{4} \int_{0}^{\sqrt{y}} yx \, dx \, dy
\]

\[
= \int_{0}^{2} \frac{y^{2}}{2} \bigg|_{0}^{2} dy = \int_{0}^{2} y^{2} \, dy = \left( \frac{y^{3}}{3} \right) \bigg|_{0}^{2} = \frac{32}{3}
\]