Lecture 14 - Polar Calculus & Multiple Integrals

Note Title

Polar Calculus: tangent lines, arc length $\Big\}$ parametric

area ← postpone

Given $r = (\theta)$, can get a parametric equation:

$$x = r\cos\theta = r(\theta)\cos\theta$$

$$y = r\sin\theta = r(\theta)\sin\theta$$

We know how to do calculus with these! (θ is our parameter)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

Ex: $r = \cos\theta \Rightarrow r'(\theta) = -\sin\theta$

$$dy/dx = \frac{(-\sin\theta)\sin\theta + (\cos\theta)\cos\theta}{(-\sin\theta)\cos\theta - (\cos\theta)\sin\theta} = \frac{\cos^2\theta - \sin^2\theta}{-2\sin\theta\cos\theta} = -\frac{1}{2}\cot(2\theta)$$

Tangent line has slope 0 at $\theta = \pi/4, 3\pi/4$, etc.

Ex: $r = 2 - 2\sin\theta$ $\quad r'(\theta) = -2\cos\theta$

$$\frac{dy}{dx} = \frac{(-2\cos\theta)\sin\theta + (2-2\sin\theta)\cos\theta}{(-2\cos\theta)\cos\theta - (2-2\sin\theta)\sin\theta} = \frac{2\cos\theta(1-2\sin\theta)}{(2\sin^2\theta - 2\cos^2\theta) - 2\sin\theta}$$

$\cos\theta = 0 \Rightarrow \theta = \pi/2, 3\pi/2$ $\Big\}$ does the denom ever vanish?

$\sin\theta = 1/2 \Rightarrow \pi/6, 5\pi/6$ $\Big\}$ Yes! at $\theta = \pi/2$.

Arc length also uses the parametric form:

$$x(\theta) = r(\theta)\cos\theta \Rightarrow dx = (r'(\theta)\cos\theta - r(\theta)\sin\theta)\,d\theta$$

$$y(\theta) = r(\theta)\sin\theta \Rightarrow dy = (r'(\theta)\sin\theta + r(\theta)\cos\theta)\,d\theta$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2}\,d\theta$$

$$= \sqrt{(r')^2\cos^2\theta - \cancel{2rr'\cos\theta\sin\theta} + r^2\sin^2\theta + (r')^2\sin^2\theta + \cancel{2rr'\sin\theta\cos\theta} + r^2\cos^2\theta}\,d\theta$$

$$= \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

So the arc length between $\theta = \alpha$ and $\theta = \beta$ is

$$s = \int_\alpha^\beta \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Ex: $r(\theta) = \theta^2$ $\quad 0 \le \theta \le \pi$

$r'(\theta) = 2\theta$

$$ds = \sqrt{r(\theta)^2 + (r'(\theta))^2}\, d\theta = \sqrt{\theta^4 + 4\theta^2}\, d\theta = \theta\sqrt{\theta^2 + 4}\, d\theta$$

$$s = \int_0^\pi \theta\sqrt{\theta^2+4}\, d\theta \qquad \begin{array}{l} u = \theta^2 + 4 \\ du = 2\theta\, d\theta \end{array} \qquad \begin{array}{l} \theta = 0 \Rightarrow u = 4 \\ \theta = \pi \Rightarrow u = \pi^2 + 4 \end{array}$$

$$= \frac{1}{2}\int_4^{\pi^2+4} \sqrt{u}\, du = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{3/2}\Big]_4^{\pi^2+4} = \boxed{\frac{(\pi^2+4)^{3/2}}{3} - \frac{8}{3}}$$

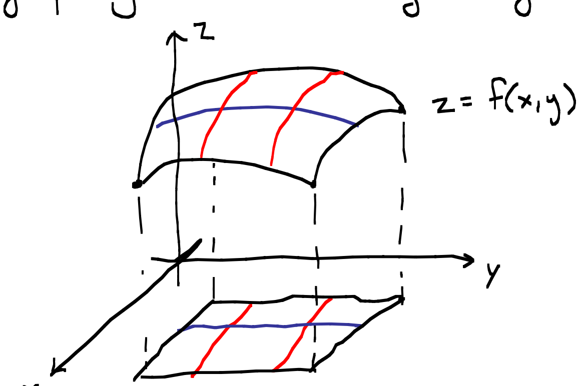
Multiple Integrals (Chapter 15: online)

Can use same ideas to address functions of 2 variables $f(x,y)$.

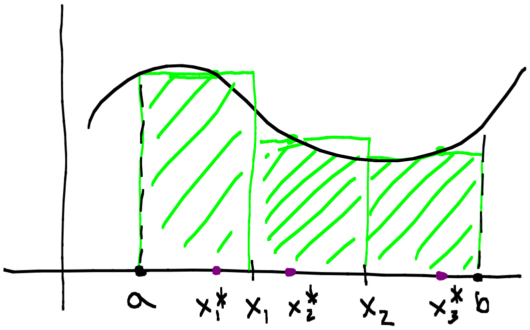
Here the domain is a subset of the (x,y)-plane, and we can do the usual Riemann Sum tricks.

Ex: $f(x,y) =$ temp or height at $x°$ long, $y°$ lat

1st: graphing: $z = f(x,y)$ gives a surface in space



Quick review of 1-var Riemann Sums:



To find area, subdivide $[a,b]$

$$x_0 = a < x_1 < x_2 < \dots < x_{k-1} < b = x_k$$

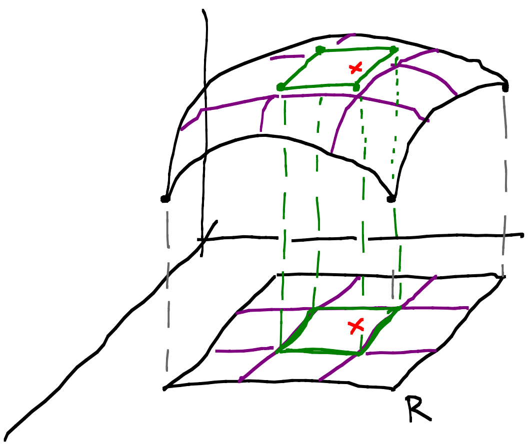
$$\Delta x_i = x_i - x_{i-1}$$

$$x_{i-1} \le x_i^* \le x_i$$

Then Area $\approx \displaystyle\sum_{i=1}^{k} f(x_i^*)\,\Delta x_i$;

<u>Def</u> $\displaystyle\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{k} f(x_i^*)\,\Delta x_i$.

Now the 2 variable case: want to find <u>volume</u> under $z = f(x,y)$



Break $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$ into sub rectangles & look at a box over each rectangle of height $f(x_i^*, y_j^*)$.

Then volume is $\approx$

$$\sum f(x_i^*, y_j^*) \cdot \underbrace{\text{area of small rectangle}}_{\Delta A}$$

$$\Delta A = \Delta x_i \cdot \Delta y_j$$

x-subdivision y-subdivision

<u>Def</u> $\displaystyle\iint_R f(x,y)\,dA = \lim_{\Delta A \to 0} \sum f(x_i^*, y_j^*)\,\Delta A$

a priori, this has nothing to do with 2 integrals. $\displaystyle\iint_R \ dA$ is a fixed symbol, defined by the Riemann sum.